ZIME Results and Solutions

Ze Committee

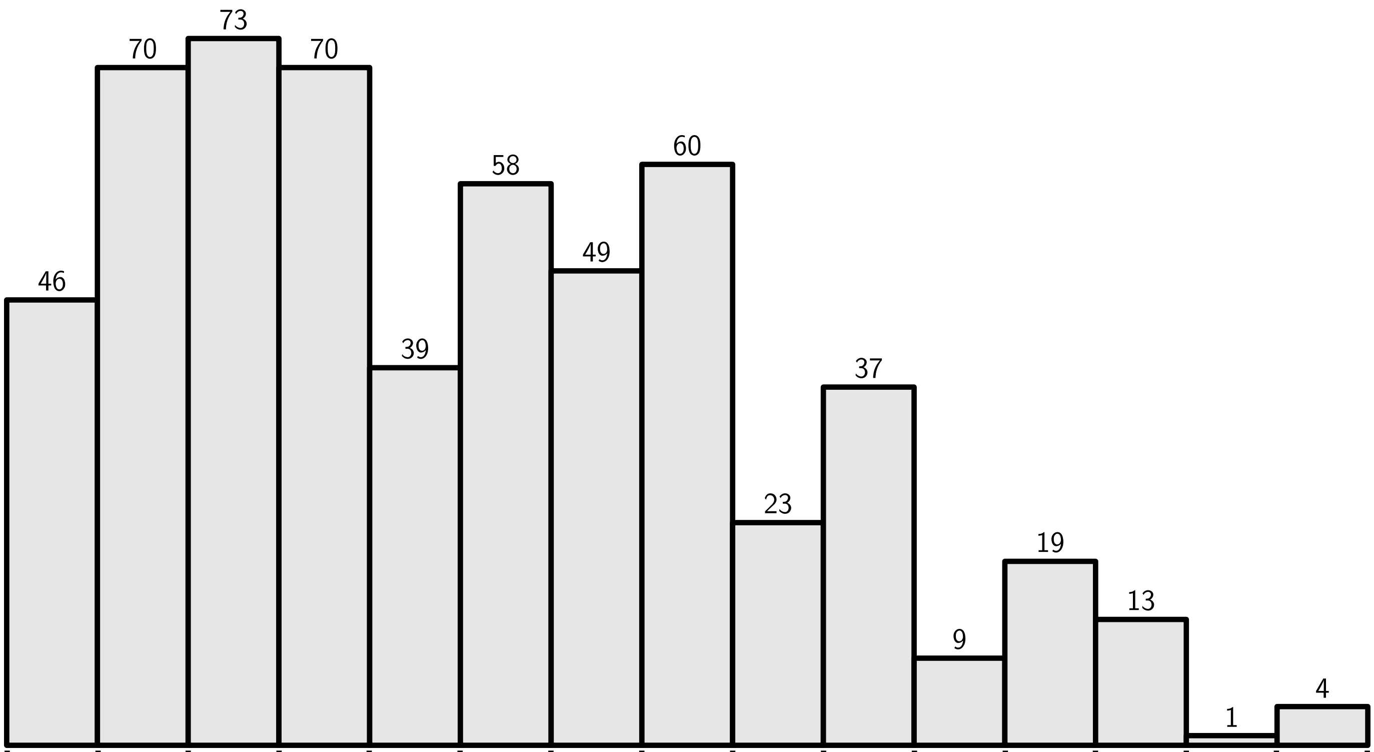
Tuesday, February 20th

This document contains results and solutions for the ZIME, a mock AIME held for the 2023-2024 competition season. The test can be found here. If you have any questions, you should contact ihatemath123 on AoPS, or bennywang on Discord. You can also check the ZeMC public discussion forum.

The ZeMC competition series is made possible by the contributions of the following problem-writers and test-solvers:

asbodke, ayush_agarwal, bissue, Geometry285, ihatemath123, kante314, OronSH, peace09, P_Groudon, Significant and Turtwig113

Answer Key and Solve Rates



1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
418	750	520	504	882	115	263	230	649	599	546	066	318	068	135

Problem 1:

If two palindromes (numbers which read the same backwards and forwards) sum to 2024, find the sum of all possible values for the smaller palindrome.

Proposed by ihatemath123.

We will use casework on the number of digits in each palindrome:

- Both four digits. Since each palindrome is at least 1000, each palindrome is at most 1024. The only palindrome in this range is 1001, and evidently $1001 + 1001 \neq 2024$, so there are no solutions in this case.
- One four-digit, one three-digit. Since the three-digit palindrome is at least 101, the four-digit palindrome is at most $2024 - 101 = 1923$; in particular, it must begin and end with a 1. So, checking the units digit, the three-digit palindrome must begin and end with a 3.

  Furthermore, it's clear by the divisibility-by-11 rule that all four-digit palindromes are multiples of 11. Since 2024 is also a multiple of 11, it follows that our three digit palindrome must be a multiple of 11. So, by the same divisibility trick, it follows that the middle digit of the three-digit palindrome is 6. Subtracting 363 from 2024 gives us our four-digit palindrome, 1661.

  So, 363 is one possibility.
- One four-digit, one two-digit. If our four-digit palindrome begins and ends with a 1, by checking the units digit, our two-digit palindrome must begin and end with a 3; in other words, it must be 33. Subtracting this from 2024 gives us our four-digit palindrome, 1991. So, 33 is one possibility.

  Or, if our four-digit palindrome begins and ends with a 2, our two-digit palindrome must be 22, so our four-digit palindrome would be 2002; we have 22 as our last possibility.
- One four-digit, one one-digit. We can run through the 9 single-digit palindromes to verify that they all fail.
- Two palindromes less than four digits long will sum to at most $999 + 999 = 1998$, so we don't have to worry about these cases.

In total, the sum of our possible palindromes is $363 + 33 + 22 = \boxed{418}$.

Problem 2:

Find the number of ways that the set $\{1, 2, 3, \ldots, 18\}$ can be split into two indistinguishable sets of nine elements, such that one set has a median of 6 and the other set has a median of 12.

Proposed by ihatemath123.

Let S be the set with median 6 and let T be the set with median 12. There must be four elements in S less than 6 and the remaining element must belong to T. There are $\binom{5}{1} = 5$ ways to partition the first 5 integers between S and T.

Next, there must be four elements in T greater than 12, and the remaining two elements must belong to S. There are $\binom{6}{2} = 15$ ways to partition the last 6 integers between S and T.

There are two more integers which must be added to S and three more integers which must be added to T. For the 5 integers between 6 and 12, there are $\binom{5}{2} = 10$ ways to partition them between S and T.

Multiplying the possiblilites in each section together gives us our total: $5 \cdot 15 \cdot 10 = \boxed{750}$.

Problem 3:

Evan thinks of two positive integers. Their quotient, which leaves no remainder, divides their sum, and their product is $784{,}000$. Find the remainder when the absolute difference between Evan's two numbers is divided by 1000.

Proposed by ihatemath123.

Let m and n be Evan's two integers. We have that $mn = 784{,}000 = 2^7 \cdot 5^3 \cdot 7^2$. We also know that $\frac{m}{n}$ is an integer, as well as

$$\frac{m+n}{\left(\frac{m}{n}\right)} = \frac{mn+n^2}{m} = n + \frac{n^2}{m}.$$

So, n divides m, which divides n^2. So:

- The exponent of 2 in the prime factorization of n must be no greater than 3, since n divides m. It must also be no less than 3, since n^2 is a multiple of m. Therefore, the exponent of 2 in the prime factorization of n is exactly 3.
- The exponent of 5 in the prime factorization of n must be no greater than 1, since n divides m. It must also be no less than 1, since n^2 is a multiple of m. Therefore, the exponent of 5 in the prime factorization of n is exactly 1.
- The exponent of 7 in the prime factorization of n must be no greater than 1, since n divides m. It must also be no less than 1, since n^2 is a multiple of m. Therefore, the exponent of 7 in the prime factorization of n is exactly 1.

So, $n = 2^3 \cdot 5 \cdot 7 = 280$ and $m = 2^4 \cdot 5^2 \cdot 7 = 2800$. Their difference is 2520 which leaves a remainder of $\boxed{520}$ when divided by 1000.

Problem 4:

Julie picks positive reals b and x with $b \neq 1$ and writes down the logarithm $\log_b x$. If she were to erase b and replace it with $\frac{b}{2}$, the value of the logarithm would increase by 12. *Instead*, if she were to erase b and replace it with $2b$, the value of the logarithm would decrease by 9. Find $\log_2(x)$.

Proposed by ihatemath123.

Let $X = \log_2 x$ (what we're asked to find) and let $B = \log_2 b$. Using the change-of-base log rule, we have

$$\begin{cases} \frac{X}{B-1} - \frac{X}{B} & = 12 \\ \frac{X}{B+1} - \frac{X}{B} & = -9. \end{cases}$$

Cross multiplying in both equations and simplifying the left hand sides gives us

$$\begin{cases} X & = 12B(B-1) \\ X & = 9B(B+1). \end{cases}$$

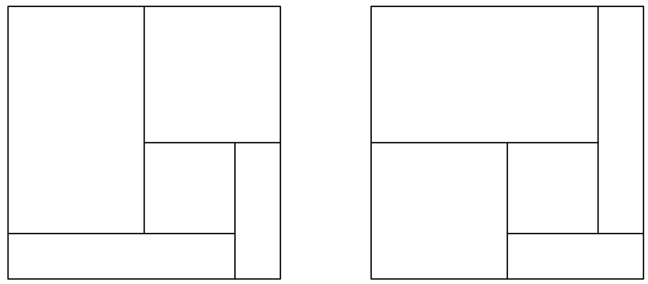
Dividing the second equation by the first gives us $\frac{B+1}{B-1} = \frac{4}{3}$, so $B = 7$. Plugging this back into either equation gives us $X = \boxed{504}$.

Problem 5:

Find the number of ways to split an eight by eight square into five rectangles with integer side lengths, no two of which share more than one vertex in common. (Rotations and reflections are considered distinct.)

Proposed by ihatemath123.

It's easy to show that the below two situations are the only ways to split a square satisfying the conditions in the problem:



It suffices to pick the central rectangle and then extend its sides in one of the two ways shown above. This central rectangle can't share an edge with the eight by eight square, so there are 7 horizontal lines to choose two horizontal sides from, as well as 7 vertical lines to choose two vertical sides from. Our final answer is

$$\left(\binom{7}{2} \cdot \binom{7}{2}\right) \cdot 2 = \boxed{882}.$$

Problem 6:

Let m and b be real numbers. Distinct points A, B, C and D lie on the line $y = mx + b$ in that order, equally spaced. Given that A and C lie on the parabola $y = x^2 + 9x + 19$ and B and D lie on the parabola $y = x^2 + x + 15$, find mb.

Proposed by ihatemath123.

We can rewrite the equations of the parabola in vertex form:

$$y = \left(x + \frac{9}{2}\right)^2 - \frac{5}{4}, \qquad y = \left(x + \frac{1}{2}\right)^2 + \frac{59}{4}.$$

So, a shift 4 units right and 16 units up will send the first parabola to the second one. This shift must also send segment $\overline{AC}$ to $\overline{BD}$, since $AC = BD$ and the two segments obviously have the same slope. So,

$$\overrightarrow{AB} = \overrightarrow{BC} = \overrightarrow{CD} = \langle 4, 16 \rangle$$

and consequently $m = \frac{16}{4} = 4$.

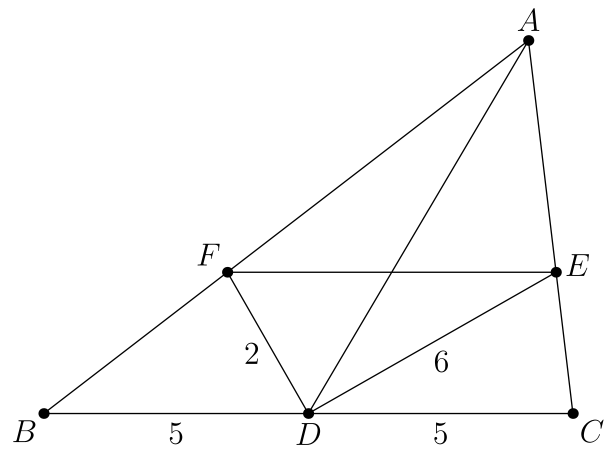
The x-coordinates of points A and C are the values of x such that $4x + b = x^2 + 9x + 19$; rearranging, we have that $x^2 + 5x + 19 - b = 0$. So, by Vieta's formula, the *sum* of the x-coordinates of A and C is -5. Since $\overrightarrow{AC} = \langle 8, 32 \rangle$, it follows that the difference in the x-coordinates is 8. So, the x-coordinates of A and C are $-\frac{13}{2}$ and $\frac{3}{2}$.

So, plugging the x-coordinate $-\frac{13}{2}$ into the equation of the first parabola, we have that $A = (-\frac{13}{2}, \frac{11}{4})$. Plugging this into the equation $y = 4x + b$ gives us that $b = \frac{115}{4}$, so $mb = 4 \cdot \frac{115}{4} = \boxed{115}$.

Problem 7:

In $\triangle ABC$, points D, E and F lie on segments $\overline{BC}$, $\overline{AC}$ and $\overline{AB}$ such that $BD = DC = 5$, $\overline{DE}$ bisects $\angle ADC$ and $\overline{DF}$ bisects $\angle ADB$. If $DE = 6$ and $DF = 2$, the length DA can be expressed as $\frac{a+\sqrt{b}}{c}$, where a, b and c are positive integers with $\gcd(a, c) = 1$. Find $a + b + c$.

Proposed by ihatemath123.



We have that

$$\angle EDF = \angle EDA + \angle FDA = \frac{1}{2} \cdot (\angle CDA + \angle BDA) = 90^\circ,$$

so $\triangle DEF$ is right. By the Pythagorean theorem, $EF = 2\sqrt{10}$. By the angle bisector theorem,

$$\frac{AF}{FB} = \frac{AD}{DB} = \frac{AD}{DC} = \frac{AE}{AC},$$

so $\overline{FE} \parallel \overline{BC}$. Using similar triangle ratios, $\frac{AF}{AB} = \frac{2\sqrt{10}}{10} = \frac{\sqrt{10}}{5}$, so $\frac{AF}{FB} = \frac{\sqrt{10}}{5-\sqrt{10}} = \frac{\sqrt{10}+2}{3}$. We now use the angle bisector theorem again to finish:

$$\frac{AD}{DB} = \frac{AF}{FB} \implies AD = \frac{AF \cdot DB}{FB} = \frac{5\sqrt{10}+10}{3}.$$

Extracting gives us our final answer of $\boxed{263}$.

Problem 8:

If a, b and c are complex numbers such that

$$\begin{cases} |a| + b + c &= 7i \\ a + |b| + c &= 9i \\ a + b + |c| &= 10i, \end{cases}$$

find $|a + b + c|^2$.

Proposed by ihatemath123.

Solution 1 (due to Eibc): We want

$$\begin{aligned} |a+b+c|^2 &= |a+b+c||\overline{a}+\overline{b}+\overline{c}| \\ &= |a|^2 + |b|^2 + |c|^2 + \sum_{\text{sym}} a\overline{b} \end{aligned}$$

However, from the first equation we have

$$\begin{aligned} 49 &= ||a| + b + c|^2 \\ &= ||a| + b + c|||a| + \overline{b} + \overline{c}| \\ &= |a|^2 + |b|^2 + |c|^2 + |a|(b + \overline{b} + c + \overline{c}) + b\overline{c} + \overline{b}c \\ &= |a|^2 + |b|^2 + |c|^2 + |a|(2\text{Re}(b + c)) + b\overline{c} + \overline{b}c. \end{aligned}$$

However, from looking at the real part of the first equation we have $\text{Re}(b + c) = -|a|$, so this is equal to

$$|a|^2 + |b|^2 + |c|^2 + |a|(-2|a|) + b\overline{c} + \overline{b}c = -|a|^2 + |b|^2 + |c|^2 + b\overline{c} + \overline{b}c.$$

Similarly, we have $|a|^2 - |b|^2 + |c|^2 + a\overline{c} + \overline{a}c = 81$ and $|a|^2 + |b|^2 - |c|^2 + a\overline{b} + \overline{a}b = 100$, and summing these three gives

$$|a|^2 + |b|^2 + |c|^2 + \sum_{\text{sym}} a\overline{b} = 49 + 81 + 100 = \boxed{230}.$$

Solution 2[1]*:* Taking the imaginary parts of both sides of each equation gives us three linear equations. We can solve this system of equations to get that $\text{Im}(a) = 6$, $\text{Im}(b) = 4$ and $\text{Im}(c) = 3$. Let x, y and z be the reals such that $a = x + 6i$, $b = y + 4i$ and $c = z + 3i$. Substituting this into our system of equations gives us

$$\begin{cases} y + z &= -|x + 6i| \\ x + z &= -|y + 4i| \\ x + y &= -|z + 3i|. \end{cases}$$

Squaring each equation, rearranging and adding all the equations together gives us

$$x^2 + y^2 + z^2 + 2xy + 2yz + 2xz = 6^2 + 4^2 + 3^2.$$

The left hand side factors as $(x + y + z)^2$. So,

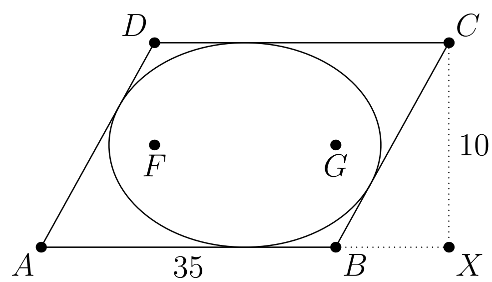
$$|a + b + c|^2 = (6 + 4 + 3)^2 + 6^2 + 4^2 + 3^2 = \boxed{230}.$$

[1] This was my original solution. I never noticed that the answer, 230, was equal to $7^2 + 9^2 + 10^2$ – I first realized this when I read Eibc's solution (solution 1) in the discussion forum. - ihatemath123

Problem 9:

Parallelogram $ABCD$ has an area of 350 and satisfies $AB = 35$. Let F and G be points in the interior of the parallelogram such that $FG = 24$ and $FG \parallel AB$. If there exists an ellipse with foci F and G tangent to all four sides of the parallelogram, find BC^2.

Proposed by OronSH.



Let X be the foot from C to line AB. The area information tells us that $CX = 10$. This is also the minor axis of the ellipse. So, we can use the well known ellipse formula

$$(\text{minor axis})^2 + (\text{dist. between foci})^2 = (\text{major axis})^2$$

to get that the major axis of this ellipse is 26 units long. Since the major axis contains $\overline{FG}$, it is parallel to $\overline{AB}$, so when we vertically stretch the figure, the major axis of the ellipse will remain parallel to $\overline{AB}$. We vertically stretch the figure by a factor of $\frac{26}{10}$, so that the minor axis of the ellipse is 26 as well, making it a circle.

This stretch preserves parallel lines, so the resulting shape is still a parallelogram. Since it has an incircle, it must be a rhombus. So, under the stretch, BC is elongated to 35 units, CX is elongated to 26 units and BX is unchanged. Since $\triangle CBX$ is still a right triangle, it follows that $BX = 35^2 - 26^2$, so

$$BC^2 = BX^2 + CX^2 = 35^2 - (10^2 + 24^2) + 10^2 = 35^2 - 24^2 = \boxed{649}.$$

Problem 10:

Alex and Oron are playing a game. They take turns spinning a fair spinner with ten sectors of equal size, numbered $1, 2, \ldots, 10$. Alex goes first. After the first spin, if a player spins a number less than or equal to the number previously spun, the game ends and the other player wins. The probability that Oron loses the game can be expressed as $\frac{m}{n}$, where m and n are coprime integers. Find the remainder when $m + n$ is divided by 1000.

Proposed by ihatemath123.

After k spins, there are $\binom{10}{k}$ possible sets of numbers which have been spun, so the probability that the game is still going after k spins is $\binom{10}{k} \cdot 10^{-k}$. So, the probability that Oron loses (i.e. the probability that the game ends after an odd number of legal spins) is

$$\begin{aligned} &\left(\binom{10}{1} \cdot 10^{-1} - \binom{10}{2} \cdot 10^{-2}\right) + \cdots + \left(\binom{10}{9} \cdot 10^{-9} - \binom{10}{10} \cdot 10^{-10}\right) \\ = \;& 1 - \left(1 - \frac{1}{10}\right)^{10} \\ = \;& \frac{10^{10} - 9^{10}}{10^{10}}. \end{aligned}$$

Computation gives us $(m + n) \bmod 1000 = \boxed{599}$.

Problem 11:

Let n and k be positive integers such that the sum of the n smallest perfect powers of k (including 1) is a multiple of 1001. Find the number of possible values of n less than 1001.
Proposed by P_Groudon

By the Chinese Remainder Theorem, we may split this up into three parts: we want to find the possible values of n in a pair (n, k) that satisfies *all three* of these equations:

$$\begin{cases} 1+k+\cdots+k^{n-1} & \equiv 0 \pmod 7 \\ 1+k+\cdots+k^{n-1} & \equiv 0 \pmod{11} \\ 1+k+\cdots+k^{n-1} & \equiv 0 \pmod{13}. \end{cases}$$

We will just work with the first one:

$$1+k+\cdots+k^{n-1} \equiv 0 \pmod 7.$$

Firstly, if $k \equiv 1 \pmod 7$, then we must have $n \equiv 0 \pmod 7$ since $1, k, \ldots, k^{n-1}$ are all $1 \pmod 7$.

If $k \not\equiv 1 \pmod 7$, we can simplify the geometric series as $\frac{k^n-1}{k-1}$. Because $k-1 \not\equiv 0 \pmod 7$, the geometric series is a multiple of 7 if and only if

$$k^n - 1 \equiv 0 \pmod 7.$$

Let p be a primitive root mod 7. From the above equation, it's clear that $k \neq 0$, so we can express k as p^e. Since we are assuming $k \neq 1$, it follows that $e \not\equiv 0 \pmod 6$. However, the equation above implies that $e \cdot n \equiv 0 \pmod 6$; if $e \not\equiv 0 \pmod 6$, then n must share *some* prime factors with 6.

From just our first equation, we know that at least one of the following is true: $n \equiv 0 \pmod 7$, $\gcd(n, 6) \neq 1$.

Similarly, the other two equations tell us that either $n \equiv 0 \pmod{11}$ or $\gcd(n, 10) \neq 1$ or both; either $n \equiv 0 \pmod{13}$ or $\gcd(n, 12) \neq 1$ or both.

Now, to count the number of such n, we split into cases:

- If $n \equiv 3 \pmod 6$, then $\gcd(n, 6) \neq 1$ and $\gcd(n, 12) \neq 1$. Therefore, the only remaining condition on n is that at least one of the following is true: $n \equiv 0 \pmod{11}$ or $\gcd(n, 10) \neq 1$. In the first case, $n \equiv 33 \pmod{66}$, which yields 15 solutions for n. In the second case, since n is odd, it must be a multiple of 5; therefore, $n \equiv 15 \pmod{30}$, which yields 33 solutions for n.

  We've overcounted the cases where n is a multiple of both 5 and 11; this is when $n \equiv 165 \pmod{330}$, which is 3 overcounted solutions.

  Therefore, this case yields $33 + 15 - 3 = 45$ solutions in total.

- If $n \equiv 0 \pmod 2$, then $\gcd(n, 6)$, $\gcd(n, 12)$ and $\gcd(n, 10)$ are all not equal to 1, so all three conditions are all immediately satisfied. So, all even n work, yielding 500 solutions.

- Otherwise, $\gcd(n, 6) = 1$. Consequently, $\gcd(n, 12) = 1$ as well. So, $n \equiv 0 \pmod 7$ and $n \equiv 0 \pmod{13}$. Because $n < 1001$, we cannot have $n \equiv 0 \pmod{11}$ as well; therefore, $\gcd(n, 10) \neq 1$. Since n is not even, it must be a multiple of 5. So, this case yields one solution: $n = 7 \cdot 13 \cdot 5 = 455$.

Altogether, we have $45 + 500 + 1 = \boxed{546}$ solutions.

Problem 12:

Points D and E lie on sides AB and AC of $\triangle ABC$, respectively, such that the circumcircles ω_1 and ω_2 of $\triangle ABE$ and $\triangle ADC$, respectively, meet on side $\overline{BC}$. Line DE meets ω_1 and ω_2 at points X and Y, respectively, such that $BC = 14$, $XY = 19$, $BD = 6$ and $CE = 9$. If the length DE can be expressed as $\frac{a-\sqrt{b}}{c}$, where a, b and c are positive integers with $\gcd(a, c) = 1$, find $a + b + c$.

Proposed by bissue.

Let F be the point on $\overline{BC}$ where ω_1 and ω_2 intersect. Now we have that

$$\begin{aligned}
DE(19 - DE) &= DE(XD + EY) \\
&= DE \cdot XD + DE \cdot EY \\
&= BD \cdot DA + CE \cdot EA \\
&= BD \cdot (BA - BD) + CE \cdot (CA - CE) \\
&= (BD \cdot BA - BD^2) + (CE \cdot CA - CE^2) \\
&= BD \cdot BA + CE \cdot CA - 36 - 81 \\
&= BF \cdot BC + CF \cdot CB - 36 - 81 \\
&= BC \cdot (BF + FC) - 36 - 81 \\
&= BC^2 - 36 - 81 \\
&= 196 - 36 - 81 \\
&= 79.
\end{aligned}$$

So, we can solve a quadratic to get that $DE = \frac{19 \pm \sqrt{45}}{2}$. Extracting gives us $19 + 45 + 2 = \boxed{066}$.

Problem 13:

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers such that $a_1 = 20$, $b_1 = 23$ and

$$\begin{cases} a_{i+1} &= \sqrt{|a_i \cdot b_i|} + \frac{a_i+b_i}{2} \\ b_{i+1} &= \sqrt{|a_i \cdot b_i|} - \frac{a_i+b_i}{2} \end{cases}$$

for all positive integers i. Find the smallest integer k for which $a_k = b_k$.

Proposed by ihatemath123.

Our recurrence is homogeneous, so for simplicity we multiply the initial terms by 2: $a_1 = 40$ and $b_1 = 46$.

If $a_k = b_k$, it follows that $a_{k-1} + b_{k-1} = 0$, from which it follows that $a_{k-2}b_{k-2} = 0$. So, we want to find the smallest k such that $a_{k-2}b_{k-2} = 0$.

For all positive i, we have that

$$a_{i+2}b_{i+2} = |a_{i+1}b_{i+1}| - \left(\frac{a_i + b_i}{2}\right)^2 = |a_{i+1}b_{i+1}| - \left(\frac{2\sqrt{|a_ib_i|}}{2}\right)^2 = |a_{i+1}b_{i+1}| - |a_ib_i|.$$

Letting $c_i = |a_ib_i|$, the recurrence becomes

$$c_{i+2} = |c_{i+1} - c_i|.$$

We can manually compute that $c_1 = 1840$ and $c_2 = 9$. Now, we just have to repeat this recurrence until we reach a term c_i which equals 0. The first few terms look like this:

$$1840, 9, 1831, 1822, 9, 1813, 1804, 9, 1795, 1786, 9, \ldots$$

We have that $c_{3i-2} = 1858 - 18i$, $c_{3i-1} = 9$ and $c_{3i} = 1849 - 18i$ for $1 \leq i \leq 102$. So, we compute the terms from c_{307} onwards. We know that $c_{305} = 9$ and $c_{306} = 13$, so we continue from c_{307}:

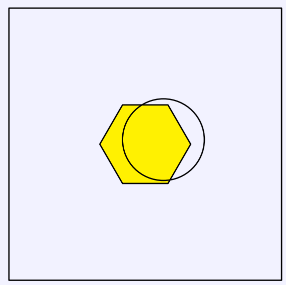
$$4, 9, 5, 4, 1, 3, 2, 1, 1, 0.$$

So, $c_{316} = 0$, from which it follows that $a_{318} = b_{318}$. Therefore, our answer is $\boxed{318}$.

Problem 14:

Alexandre forms a piece of cookie dough in the shape of a regular hexagon with a side length of 8 cm, and places it at the center of a square baking pan with a side length of 50 cm, as shown in the diagram below. He then drops a circular cookie cutter with a radius of 7 cm randomly and uniformly inside the baking pan, such that the entire cutter lies within the pan. The expected number of pieces that the cookie dough gets cut into can be expressed as $\frac{m}{n}$ for coprime positive integers m and n. Find $m+n$.

(For example, in the diagram below, the cookie is cut into two pieces. If the cookie cutter does not touch the dough, the cookie dough is in 1 piece.)



Proposed by ihatemath123.

The probability that the cookie cutter's circle is tangent to the perimeter of the cookie is 0, so we can disregard this case.

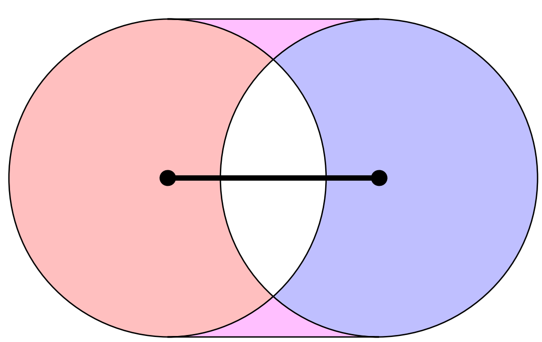
Claim: We have

$$(\#\text{ pieces}) = \frac{(\#\text{ intersections})}{2} + 1.$$

The cookie cutter will split the cookie into some pieces which lie outside of the cookie cutter and a single piece which lies inside the cookie cutter. For the pieces which lie outside of the cutter, along its perimeter, there will be two intersections between the cutter and the cookie. Therefore, for each piece of the cookie which lies outside the cutter, there are two corresponding intersections. Adding the single piece of the cookie which lies inside the cutter gives us our desired claim. □

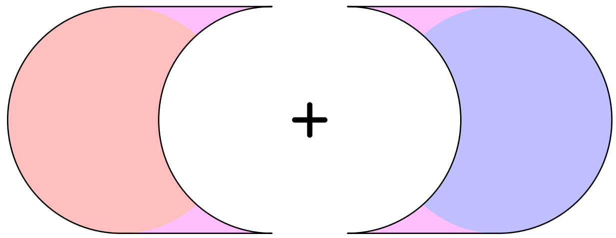
There are two ways to proceed:

Solution 1A: It suffices to calculate the expected number of intersections between the cookie cutter and one side of the hexagon, since because of linearity of expectation, we can multiply that value by 6.



In the diagram above, a randomly dropped circle hits the segment exactly once if it lands in the red or blue region, and hits the segment twice if it lands in the pink region. It suffices to find

$$[\text{red}] + [\text{blue}] + 2 \cdot [\text{pink}] = ([\text{red}] + [\text{pink}]) + ([\text{blue}] + [\text{pink}]) =$$



$$= 8 \cdot 14 + 8 \cdot 14 = 224.$$

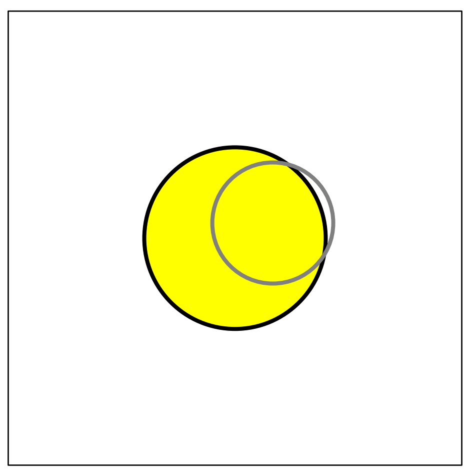
In particular, on the last step, we can calculate the area of the red+pink and blue+pink shapes easily because they are just skewed rectangles with a base of 8 and a height of $2 \cdot 7 = 14$.

So, the expected number of intersections between the cookie cutter and a side of the hexagon is $\frac{224}{36^2}$. To finish, the expected number of pieces is

$$\frac{6 \cdot \frac{224}{36^2}}{2} + 1 = \frac{41}{27} \implies \boxed{068}.$$

Solution 1B: Because we're calculating the expected number of *intersections* between the cookie and the cutter, linearity of expectation "tells us" that the shapes of the cookie and the cutter don't actually matter – this expected value that we're searching for *only depends on their perimeters*.

So, instead of a hexagonal cookie, we will assume that it is a circle with radius $\frac{24}{\pi}$. Then, the cutter's circle is tangent to the cookie's circle with probability 0; otherwise, they intersect at 0 or 2 points. The circles intersect at two points if and only if the center of the cookie cutter lies within 7 units of the circumference of the cookie.



So, the success region is an annulus with an inner radius of $\frac{24}{\pi} - 7$ and an outer radius of $\frac{24}{\pi} + 7$. The area of the annulus is

$$\pi \cdot \left(\left(\frac{24}{\pi} + 7 \right)^2 - \left(\frac{24}{\pi} - 7 \right)^2 \right) = 4\pi \cdot \frac{24}{\pi} \cdot 7 = 672,$$

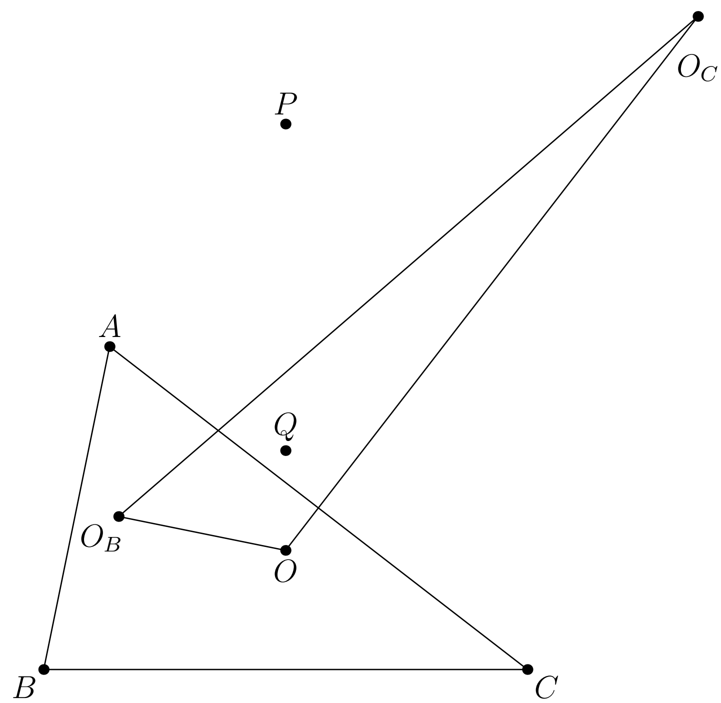
so the expected number of intersections is $\frac{672 \cdot 2}{36^2}$. Finally, the expected number of pieces is

$$\frac{672 \cdot 2}{36^2 \cdot 2} + 1 = \frac{41}{27} \implies \boxed{068}.$$

Problem 15:

Let $\triangle ABC$ be an acute triangle with circumcenter O. Let O_B and O_C be the circumcenters of $\triangle BOA$ and $\triangle COA$, respectively, and let P be the circumcenter of $\triangle OO_BO_C$. If the circumradii of $\triangle ABC$, $\triangle OO_BO_C$ and $\triangle PBC$ are 9, 15 and 11, respectively, find AP^2.

Proposed by ihatemath123.



Claim: Point P lies on the perpendicular bisector of $\overline{BC}$.

Let X_A be the antipode of O with respect to (BOC); define X_B and X_C similarly. Then, since $\angle OAX_B = \angle OAX_C = 90^\circ$ and likewise for other vertices of $\triangle ABC$, the intouch triangle of $\triangle X_AX_BX_C$ is $\triangle ABC$. In particular, O is the incenter of $\triangle X_AX_BX_C$.

The incenter-excenter lemma tells us that the circumcenter of $\triangle X_BOX_C$ lies on line X_AO, the perpendicular bisector of $\overline{BC}$. Now, taking a homothety with scale factor $\frac{1}{2}$ centered at O sends X_B and X_C to O_C and O_B, respectively, as well as sending the circumcenter of $\triangle X_BOX_C$ to P, the circumcenter of $\triangle O_COO_B$. This homothety sends the perpendicular bisector of $\overline{BC}$ to itself, so it follows that P lies on the perpendicular bisector of $\overline{BC}$. □

As a consequence, $\triangle PBC$ is isosceles, so Q also lies on the perpendicular bisector of $\overline{BC}$.

We now know the side lengths of $\triangle OQC \cong \triangle OQB$: we have that $QC = 11$, $OC = 9$ and $OQ = OP - QP = 4$. So, by Heron's formula, $[OQC] = 12\sqrt{2}$, so the length of the altitude from C to line OQ is $6\sqrt{2}$. Thus, $BC = 12\sqrt{2}$.

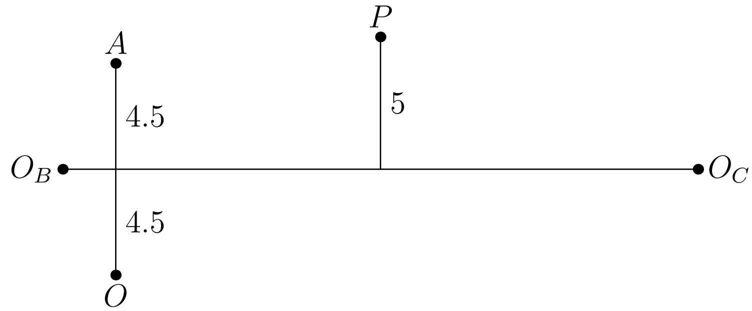
Claim: We have $\triangle BOC \sim \triangle O_BPO_C$.

Clearly, both triangles are isosceles, so it suffices to show that $\angle BOC = \angle O_BPO_C$. Let M and N be the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Then, we have that

$$\angle BOC = 2\angle A = 2 \cdot (180^\circ - \angle MON) = 2 \cdot (180^\circ - \angle O_BOO_C) = \angle O_BPO_C. \qquad \square$$

So, $O_BO_C = BC \cdot \frac{15}{9} = 20\sqrt{2}$.

Now, to finish, we "coordinate-bash", setting $\overline{O_BO_C}$ as our horizontal axis:



We can use the Pythagorean theorem to get that the distance from P to $\overline{O_BO_C}$ is 5. Furthermore, $OA = 9$ (since it is a radius of the circumcircle of $\triangle ABC$), and by definition O_B and O_C lie on the perpendicular bisector of $\overline{OA}$. Hence, segment $\overline{OA}$ is split into two segments of length 4.5.

The "vertical" distance between O and P is 9.5. Furthermore, $OP = 15$ because it is a radius of the circumcircle of $\triangle O_BOO_C$. So, the horizontal distance between O and P is $\sqrt{15^2 - 9.5^2}$.

The vertical distance between A and P is $5 - 4.5 = 0.5$; the horizontal distance is $\sqrt{15^2 - 9.5^2}$. Therefore,

$$AP^2 = 0.5^2 + 15^2 - 9.5^2 = \boxed{135}.$$