ZeMC 10 Results and Solutions

Ze Committee

Saturday, November 11th

This document contains results and solutions for the ZeMC 10, a mock AMC 10 held for the 2023-2024 competition season. The test can be found here. If you have any questions, you should contact ihatemath123 on AoPS, or bennywang on Discord. You can also check the ZeMC public discussion forum.

The ZeMC competition series is made possible by the contributions of the following problem-writers and test-solvers:

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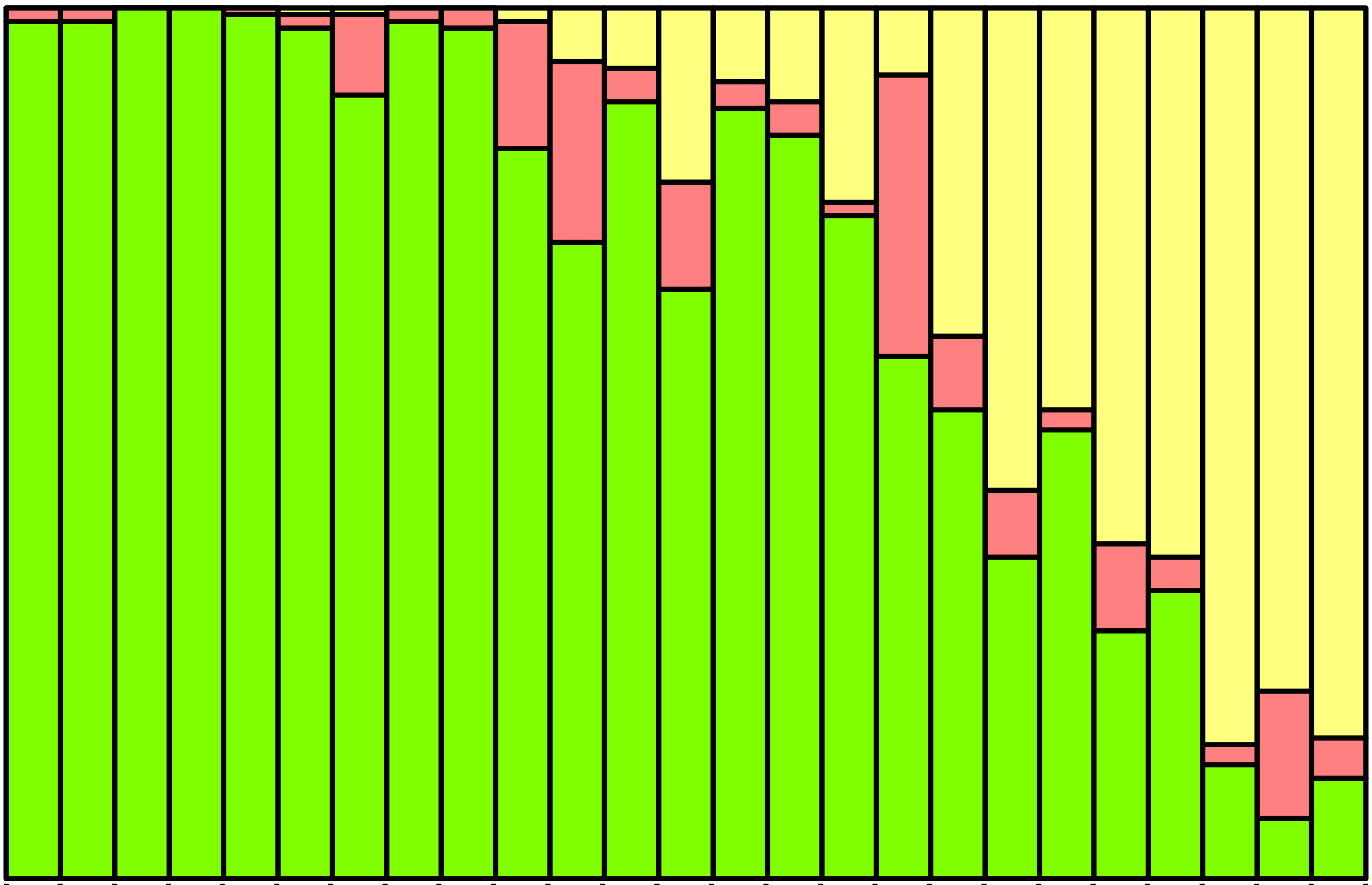
This solutions manual was written by

P_Groudon.

Answer Key and Solve Rates

The chart of solve rates below reflects the number of correct, incorrect and blank answers given for each problem on the contest. In the chart, green denotes a correct answer, red denotes an incorrect answer and yellow denotes no answer.



1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
D	D	C	D	B	C	D	C	B	A	D	B	D	E	C	A	D	E	C	C	A	D	D	E	A

Problem 1:

What is the value of
$$3.1 - 3.14 + 3.141 - 3.1415 + 3.14159?$$

(A) -0.10109 **(B)** 0.10109 **(C)** 0.14159 **(D)** 3.10109 **(E)** 3.14159

Proposed by ihatemath123.

By grouping terms, we have

$$3.1 + (-3.14 + 3.141) + (-3.1415 + 3.14159) = 3.1 + 0.001 + 0.00009 = \boxed{\textbf{(D) } 3.10109}.$$

Problem 2:

The captain of a ship has no coins. After he steals one coin from each of his ten crewmates, everybody on board has the same number of coins. How many coins are on the ship in total?

(A) 90 **(B)** 99 **(C)** 100 **(D)** 110 **(E)** 111

Proposed by ihatemath123.

Since the captain steals one coin from each of 10 crewmates, he now has 10 coins. This is the number of coins each of the 11 people on board have, so there are $11 \cdot 10 = \boxed{\textbf{D } 110}$ coins on board.

Problem 3:

Alexa evaluates the sum of a positive integer and its reciprocal. She then writes the result down as a simplified fraction. If the numerator of the fraction is 50, what is its denominator?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Proposed by ihatemath123.

Let x be Alexa's positive integer. Then, her sum is $x + \frac{1}{x} = \frac{x^2+1}{x}$. Using the Euclidean Algorithm,

$$\gcd(x^2+1, x) = \gcd(x^2+1-x \cdot x, x) = \gcd(1, x) = 1.$$

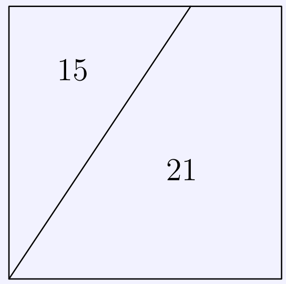
Thus, the fraction is already in lowest terms, regardless of the value of x. Since the numerator is 50,

$$x^2 + 1 = 50 \implies x = 7.$$

Thus, the denominator of the fraction is $\boxed{\textbf{(C) } 7}$.

Problem 4:

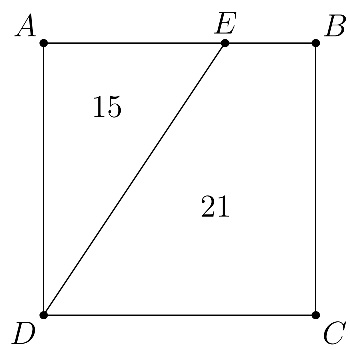
A segment drawn in a square splits it into a triangle with an area of 15 and a trapezoid with an area of 21. What is the length of this segment?



(A) $\sqrt{34}$ **(B)** 6 **(C)** $2\sqrt{13}$ **(D)** $\sqrt{61}$ **(E)** 8

Proposed by ihatemath123.

Point labels are shown below:



Clearly, $[ABCD] = [AED] + [EBCD] = 36$. Thus, the side length of the square is 6. The area of $\triangle AED$ is 15, so

$$\frac{1}{2} \cdot AD \cdot AE = 15 \implies AE = 5.$$

Then, by Pythagorean Theorem on $\triangle AED$,

$$ED = \sqrt{6^2 + 5^2} = \boxed{\textbf{(D) } \sqrt{61}}.$$

Problem 5:

Two fifths of the marbles in a bag are cyan and the rest are navy. Sara adds 30 navy marbles to the bag, so that now one fifth of the marbles in the bag are cyan. How many cyan marbles are in the bag?

(A) 10 **(B)** 12 **(C)** 15 **(D)** 18 **(E)** 20

Proposed by ihatemath123.

The fraction of marbles which were cyan halved after Sara added the navy marbles. Thus, Sara must have doubled the total number of marbles in the bag. She added 30 marbles, so there must have been 30 marbles initially. Two fifths of these were cyan, which is $\boxed{\textbf{(B) } 12}$.

Problem 6:

How many of the following statements are true?

(a) Statement (b) is false.

(b) Statement (c) is false.

(c) Statement (d) is false.

(d) Statement (a) is false.

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Proposed by ihatemath123.

For any two consecutive statements, one is true while the other is false. So, the statements are either TFTF or FTFT, both of which can be checked to work. Either way, the number of true statements is $\boxed{\textbf{(C) } 2}$.

Problem 7:

The square of Jason's favorite integer begins with a 4. What is the sum of all possible digits that Jason's favorite integer could begin with?

(A) 9 **(B)** 11 **(C)** 13 **(D)** 15 **(E)** 17

Proposed by ihatemath123.

When we square root a number which begins with 4, we will get a possibility for Jason's number. The square root of a number between 4 and 5 must begin with a 2, while the square root of a number between 40 and 50 must begin with a 6 or a 7. We don't need to the square root of a number between 400 or 500, or anything beyond that, since that square root will be 10 times a smaller number we've already checked. (For example, $\sqrt{1234} = 10 \cdot \sqrt{12.34}$.)

Jason's number must begin with a 2, 6 or 7. For example, it could be 20, 64 or 70, which, when squared yield 400, 4096 and 4900 respectively. Our final answer is $2 + 6 + 7 = \boxed{\textbf{(D) } 15}$.

Problem 8:

The sum of an integer N and the remainder when N is divided by 23 is 300. What is the sum of the digits of N?

(A) 16 **(B)** 17 **(C)** 18 **(D)** 19 **(E)** 20

Proposed by ihatemath123.

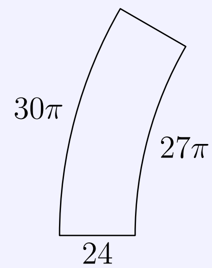
Let r be the remainder when N is divided by 23 so that $N \equiv r \pmod{23}$ and $N + r = 300$. Taking the second equation modulo 23 gives us

$$2r \equiv 1 \pmod{23} \implies r \equiv 12 \pmod{23}.$$

With $0 \leq r \leq 22$, this implies $r = 12$ and $N = 300 - 12 = 288$. Indeed, this value of N satisfies all necessary requirements. The digit sum is $\boxed{\textbf{(C) } 18}$.

Problem 9:

Victor's model train rolls on tracks, which are constructed from pieces. Each piece is shaped like an annulus sector with a width of 24 cm, an inner circumference of 27π cm and an outer circumference of 30π cm. How many pieces must Victor join together to form a full circle?



(A) 15 **(B)** 16 **(C)** 18 **(D)** 20 **(E)** 24

Proposed by ihatemath123.

Let n be the number of these pieces in a circular track. The inner circumference of the whole track would be $n \cdot 27\pi$, while the outer circumference would be $n \cdot 30\pi$. We can get the radius of these two circles by dividing by 2π, which gives us $n \cdot 13.5$ and $n \cdot 15$ for the inner and outer tracks respectively. Since we know the track is 24 units wide, we have the equation

$$n \cdot 15 - n \cdot 13.5 = 24 \implies n = \boxed{\textbf{(B) } 16}.$$

Problem 10:

At a math summer camp, 75% of the students have exactly two roommates and 25% of the students have exactly three roommates. What is the average number of students in each room?

(A) 3.2 **(B)** 3.25 **(C)** 3.3 **(D)** 3.35 **(E)** 3.4

Proposed by ihatemath123.

Suppose there are S students in total. Then, $\frac{3S}{4}$ of them have exactly 2 roommates, while $\frac{S}{4}$ have exactly 3 roommates.

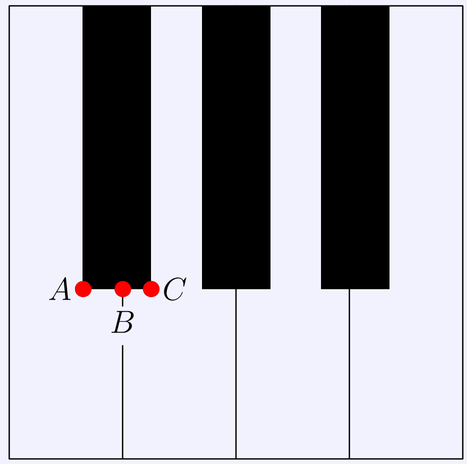
The students with exactly 2 roommates are in some room with 3 people total. Thus, these students can be divided into groups of 3 based on their room. Thus, the number of rooms with exactly 3 students is $\frac{3S}{4}/3 = \frac{S}{4}$.

The students with exactly 3 roommates are in some room with 4 people total. Thus, these students can be divided into groups of 4 based on their room. Thus, the number of rooms with exactly 4 students is $\frac{S}{4}/4 = \frac{S}{16}$.

It follows that the number of rooms with 3 students vs 4 students is a $4 : 1$ ratio. Hence, the average number of students in each room is $\frac{4}{5} \cdot 3 + \frac{1}{5} \cdot 4 = \boxed{\textbf{(A) } 3.2}$.

Problem 11:

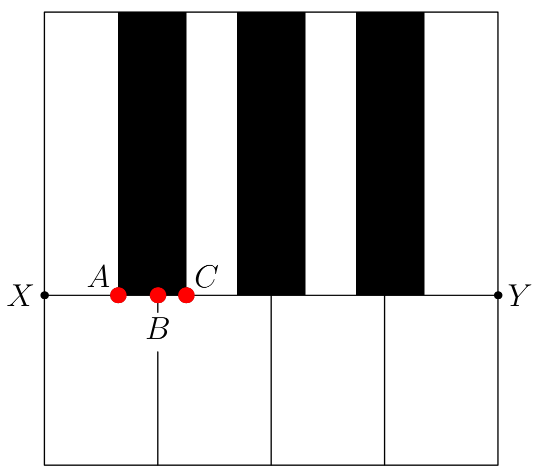
In the section of the keyboard below, each of the four white keys has the same width and area; similarly, each of the three black keys has the same width and area. What is the ratio $AB : BC$?



(A) $1:1$ **(B)** $3:2$ **(C)** $2:1$ **(D)** $3:1$ **(E)** $4:1$

Proposed by ihatemath123.

The black keys having the same width and area means they all have the same height. Extend line AC to the ends of the keyboard as shown below:



Because the 4 white keys all have the same width, $XB : BY = 1 : 3$. Also, the portion of each white key under $\overline{XY}$ is of the same area.

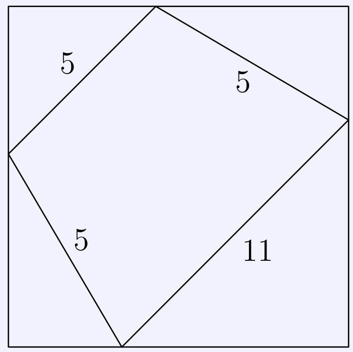
Since the white keys overall have the same area, it follows that the 4 white rectangles above $\overline{XY}$ have the same area and width, too.

Let w be the width of a white rectangle above $\overline{XY}$. Similarly, let b be the width of a black rectangle. Then, $XY = 4w + 3b$. Since $XB : BY$, it follows that $XB = w + \frac{3b}{4}$.

Clearly, $XA = w$ and $AC = b$. Thus, $AB = XB - XA = \frac{3b}{4}$, implying that $AB : BC = \boxed{\textbf{(D) } 3:1}$.

Problem 12:

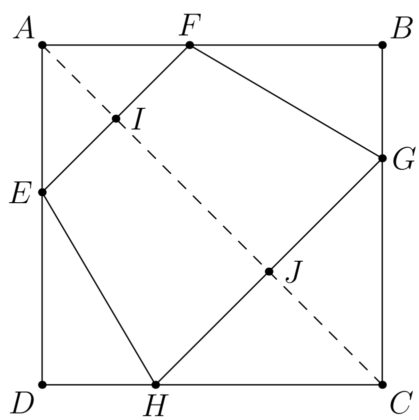
An isosceles trapezoid, with side lengths shown below, is inscribed in a square. What is the side length of the square?



(A) 8 **(B)** $6\sqrt{2}$ **(C)** 9 **(D)** $7\sqrt{2}$ **(E)** 10

Proposed by ihatemath123.

Point labels are shown below with a diagonal of the square drawn:



By symmetry, trapezoid $EFGH$ is symmetric across AC. Clearly, $\triangle EFA$ and $\triangle GHC$ are 45-45-90 triangles.

With $EF = 5$, it follows that $AI = \frac{5}{2}$. With $GH = 11$, it follows that $JC = \frac{11}{2}$.

Note that IJ is the height of trapezoid $EFGH$. Let K be the foot of the altitude from E to HG. Then, $EH = 5$ and $HK = 3$ (by symmetry of the isosceles trapezoid). Then, by Pythagorean Theorem, $EK = 4$, so $IJ = 4$.

So $AC = AI + IJ + JC = 12$. Thus, $AB = \frac{1}{\sqrt{2}} \cdot 12 = 6\sqrt{2}$. So the side length of the square is $\boxed{\textbf{(B) } 6\sqrt{2}}$.

Problem 13:

Call a day *unlucky* if it is a Friday that falls on the 13th day of the month. Which of the following is the closest to the number of *unlucky* days that have occurred after the start of the year 1900 and before the end of the year 1999?

(A) 110 **(B)** 130 **(C)** 150 **(D)** 170 **(E)** 190

Proposed by ihatemath123.

Since there are 12 months per year, there were $12 \cdot 100 = 1200$ months from 1900 to 1999, inclusive. Each month has a 13th day, so there were 1200 days that were the 13th of a month during that time frame.

By symmetry, any given day is equally likely to be any of the 7 days of the week. Hence, the number of these days that were on a Friday is approximately $1200 \cdot \frac{1}{7} \approx 171$. The closest choice is $\boxed{\textbf{(D)}\ 170}$. [1]

Problem 14:

Let $a_1, a_2, a_3, \ldots, a_{32}$ be a geometric series. If $a_1+a_2+a_3+\cdots+a_{32} = 20$ and $a_2+a_4+a_6+\cdots+a_{32} = 6$, what is $a_4 + a_8 + a_{12} + \cdots + a_{32}$?

(A) $\dfrac{19}{21}$ **(B)** $\dfrac{21}{23}$ **(C)** $\dfrac{23}{25}$ **(D)** $\dfrac{25}{27}$ **(E)** $\dfrac{27}{29}$

Proposed by ihatemath123.

Let r be the common ratio of the geometric series so that $a_{n+1} = a_n r$ for all integers $1 \le n \le 31$. Subtracting the second equation from the first,

$$a_1 + a_3 + a_5 + \cdots + a_{31} = 14.$$

Hence,

$$\frac{6}{14} = \frac{a_2 + a_4 + a_6 + \cdots + a_{32}}{a_1 + a_3 + a_5 + \cdots + a_{31}} = \frac{a_1 r + a_3 r + a_5 r + \cdots + a_{31} r}{a_1 + a_3 + a_5 + \cdots + a_{31}} = \frac{r(a_1 + a_3 + a_5 + \cdots + a_{31})}{a_1 + a_3 + a_5 + \cdots + a_{31}} = r.$$

Thus, $r = \frac{3}{7}$. Similarly,

$$\frac{a_4 + a_8 + a_{12} + \cdots + a_{32}}{a_2 + a_6 + a_{10} + \cdots + a_{30}} = \frac{a_2 r^2 + a_6 r^2 + a_{10} r^2 + \cdots + a_{30} r^2}{a_2 + a_6 + a_{10} + \cdots + a_{30}} = r^2 = \frac{9}{49}.$$

Since

$$(a_4 + a_8 + a_{12} + \cdots + a_{32}) + (a_2 + a_6 + a_{10} + \cdots + a_{30}) = 6,$$

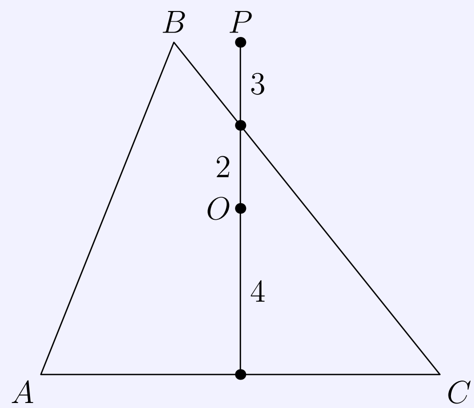
it follows that

$$a_4 + a_8 + \cdots + a_{32} = \frac{9}{49+9} \cdot 6 = \boxed{\textbf{(E)}\ \frac{27}{29}}.$$

[1] In fact, the exact answer is 171 on the dot!

Problem 15:

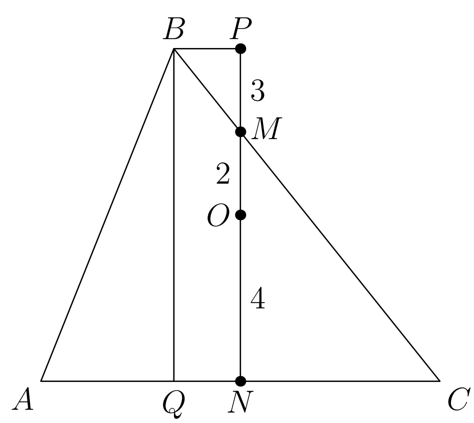
In $\triangle ABC$, the perpendicular bisector of $\overline{AC}$ is drawn from the midpoint of $\overline{AC}$ to the point P with $\overline{BP} \parallel \overline{AC}$. This segment is split into three pieces by the circumcenter and side $\overline{BC}$, with lengths that measure 4, 2 and 3. What is the area of $\triangle ABC$?



(A) 27 **(B)** $9\sqrt{10}$ **(C)** $18\sqrt{3}$ **(D)** 36 **(E)** $18\sqrt{5}$

Proposed by ihatemath123.

New points are defined below, where N is the midpoint of $\overline{AC}$, M is the intersection of NP and BC, and Q is the foot of the altitude from B to AC.



$BPNQ$ is a rectangle because opposite sides are parallel by construction and the angles are all $90°$.

Let $BP = k$. Because $\overline{BP} \parallel \overline{NC}$, it follows that $\triangle BPM \sim \triangle CNM$. Thus, $\frac{BP}{CN} = \frac{PM}{NM} = \frac{3}{6}$. Thus, $CN = 2k$. Since N is the midpoint of $\overline{AC}$, $AC = 4k$. Also, $BQ = PN = 9$ because $BPNQ$ is a rectangle. Thus, $[ABC] = \frac{1}{2} \cdot AC \cdot BQ = 18k$.

By definition of the circumcenter, $BO = OC$. Using Pythagorean Theorem on $\triangle BPO$ and $\triangle CNO$,

$$BO^2 = BP^2 + PO^2 = k^2 + 25$$

$$OC^2 = ON^2 + NC^2 = 16 + 4k^2.$$

Thus,

$$k^2 + 25 = 16 + 4k^2 \implies k = \sqrt{3} \implies [ABC] = \boxed{\textbf{(C) } 18\sqrt{3}}.$$

Problem 16:

A list of 25 consecutive positive perfect cubes and a list of k consecutive positive perfect squares both begin with the same value and end with the same value. What is the largest possible value of k?

(A) 219 **(B)** 254 **(C)** 263 **(D)** 284 **(E)** 342

Proposed by ihatemath123.

Because the two lists have the same first value, this first value is both a perfect cube and perfect square so it can be expressed as m^6 for some positive integer m. Similarly, let the last value be n^6 for some positive integer n.

The perfect cubes begin from m^2 cubed and go up to n^2 cubed – there are $n^2 - m^2 + 1$ values in this list. Setting this equal to 25, we can manipulate the expression to get a difference of squares:

$$(n-m)(n+m) = 24 \implies (n,m) = (5,1),(7,5).$$

We want to maximize the number of elements in the list of perfect squares, which is equal to $k = n^3 - m^3 + 1$. This happens when $(n,m) = (7,5)$, where k evaluates to

$$7^3 - 5^3 + 1 = \boxed{\textbf{(A) } 219}.$$

Problem 17:

How many positive integers n are there such that n^n is a divisor of 144^{144}?

(A) 15 **(B)** 16 **(C)** 19 **(D)** 20 **(E)** 21

Proposed by ihatemath123.

Note that $144^{144} = (2^4 \cdot 3^2)^{144} = 2^{576} \cdot 3^{288}$. Thus, if n is divisible by any prime other than 2 or 3, n^n cannot be a divisor of 144^{144}. It follows that $n = 2^a \cdot 3^b$ for nonnegative integers a and b. Thus,

$$n^n = (2^a \cdot 3^b)^{(2^a \cdot 3^b)} = 2^{a \cdot 2^a \cdot 3^b} \cdot 3^{b \cdot 2^a \cdot 3^b},$$

which is a divisor of $2^{576} \cdot 3^{288}$ when

$$a \cdot 2^a \cdot 3^b \le 576 \quad \text{and} \quad b \cdot 2^a \cdot 3^b \le 288.$$

- If $b = 0$, then $a \cdot 2^a \le 576$. Thus, $0 \le a \le 6$, yielding 7 values of n.
- If $b = 1$, then $a \cdot 2^a \le 192$ and $2^a \le 48$. Thus, $0 \le a \le 5$, yielding 6 values of n.
- If $b = 2$, then $a \cdot 2^a \le 64$ and $2^a \le 16$. Thus, $0 \le a \le 4$, yielding 5 values of n.
- If $b = 3$, then $a \cdot 2^a \le 21$ and $2^a \le 3$. Thus, $0 \le a \le 2$, yielding 2 values of n.
- If $b \ge 4$, then $b \cdot 2^a \cdot 3^b \ge 4 \cdot 1 \cdot 81 = 324$, contradicting the second inequality.

Thus, the total number of n is $7 + 6 + 5 + 2 = \boxed{\textbf{(D) } 20}$.

Problem 18:

Let X be the number of permutations of the first 23 positive integers for which the sum of any two adjacent numbers is no more than 24. What is the sum of the digits of X?

(A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

Proposed by ihatemath123.

Using the given condition, the only number that 23 can be adjacent to is 1. Thus, 23 must be on one of the two ends of the permutation with 1 next to it. There are 2 choices to place $\{1, 23\}$ in the permutation.

Since 1 can be next to any of $2, 3, ..., 23$, the 1 and 23 can be ignored after placing them and then it remains to consider how $2, 3, ..., 22$ are permuted.

Similar to before, the only number left that 22 can be adjacent to is 2, implying that 22 is at one of the two ends of the remaining spaces with 2 next to it. Thus, there are 2 choices to place $\{2, 22\}$ in the permutation.

Since 2 can be next to any of $3, 4, ..., 22$, the 2 and 22 can be ignored after placing them and then it remains to consider how $3, 4, ..., 21$ are permuted.

Inductively, for each of

$$\{1, 23\}, \{2, 22\}, \{3, 21\}, ..., \{11, 13\},$$

there are 2 choices to place the set of numbers in the permutation after all previous choices are made. Then, 12 will be in the remaining blank space. Thus, $X = 2^{11} = 2048$, which has a digit sum of $\boxed{\textbf{(E) } 14}$.

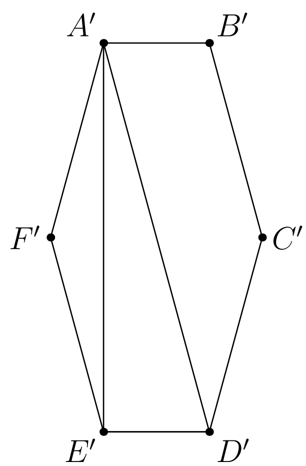
Problem 19:

Celine draws a regular hexagon $ABCDEF$ in the Cartesian plane such that $\overline{AB}$ and $\overline{DE}$ are parallel to the x-axis. She then stretches the plane vertically by a factor of some constant k and draws the image of the hexagon under this stretch, $A'B'C'D'E'F'$. If the distance between lines $A'B'$ and $D'E'$ is 5 units and the distance between lines $B'C'$ and $E'F'$ is 2 units, what is k?

(A) $\sqrt{7}$ **(B)** $\frac{8\sqrt{3}}{5}$ **(C)** $2\sqrt{2}$ **(D)** $\frac{4\sqrt{5}}{3}$ **(E)** 3

Proposed by ihatemath123.

The distance from E' to $\overline{A'D'}$ is half the distance from $\overline{F'E'}$ to $\overline{B'C'}$ (since stretching preserves $\overline{FE} \parallel \overline{AD} \parallel \overline{BC}$.) So, the distance from E' to $\overline{A'D'}$ is $\frac{2}{2} = 1$. Furthermore, $\overline{A'E'} = 5$ because it is just the distance between $\overline{A'B'}$ and $\overline{D'E'}$.



So, letting G be the foot from E' to $\overline{A'D'}$, we can calculate that

$$A'G = \sqrt{A'E'^2 - E'G^2} = \sqrt{5^2 - 1^2} = 2\sqrt{6}.$$

Because $\triangle A'GE' \sim \triangle A'E'D'$, it follows that

$$\frac{A'E'}{E'D'} = \frac{A'G}{GE'} = \frac{2\sqrt{6}}{1} = 2\sqrt{6}.$$

Prior to this vertical stretch, $\triangle AED$ was a 30-60-90 triangle (a well known property of regular hexagons). So, formerly, $\frac{AE}{ED} = \sqrt{3}$. Since it's $2\sqrt{6}$ now, we have

$$k = \frac{2\sqrt{6}}{\sqrt{3}} = \boxed{\textbf{(C) } 2\sqrt{2}}.$$

Problem 20:

Ruben writes down a system of equations. Serge spills coffee on the paper, covering a number:

$$\begin{cases} x+y+z &= 7 \\ xy+yz+xz &= 8 \\ xyz &= \blacksquare \end{cases}$$

However, Ruben remembers that there were exactly three ordered triples (x, y, z) of *positive* real numbers satisfying the system of equations. The value that should replace the $\blacksquare$ can be expressed as $\frac{m}{n}$ for coprime positive integers m and n. What is $m+n$?

(A) 91 **(B)** 93 **(C)** 95 **(D)** 97 **(E)** 99

Proposed by peace09.

Let C be the covered number. By Vieta's formulas, x, y, and z are the roots r_1, r_2, r_3 of the polynomial $P(t) = t^3 - 7t^2 + 8t - C$ in some order. Moreover, if (x, y, z) is not a permutation of (r_1, r_2, r_3), it is impossible for it to be a solution to the system of equations.

- If r_1, r_2, and r_3 are pairwise distinct, there are $3!$ solutions (x, y, z) formed by permuting (r_1, r_2, r_3).
- If $r_1 = r_2 \neq r_3$, then there are 3 solutions (x, y, z) formed by permuting (r_1, r_2, r_1).
- If $r_1 = r_2 = r_3$, there is 1 solution (x, y, z), namely (r_1, r_1, r_1).

Hence, the only way for exactly 3 positive real number triples (x, y, z) to exist is for $r_1 = r_2 \neq r_3$, with each root positive. The system of equations now becomes

$$\begin{cases} 2r_1 + r_2 &= 7 \\ 2r_1r_2 + r_1^2 &= 8 \\ r_1^2 r_2 &= C \end{cases}$$

Rewriting the first equation as $r_2 = 7 - 2r_1$ and plugging it in to the second equation implies

$$3r_1^2 - 14r_1 + 8 = 0 \implies r_1 = \frac{7 \pm 5}{3}.$$

If $r_1 = 4$, then $r_2 = -1$ by the first equation. However, this contradicts the fact that every root is positive. Thus, $r_1 = \frac{2}{3}$, implying that $r_2 = \frac{17}{3}$. Hence, $C = (\frac{2}{3})^2 \cdot \frac{17}{3} = \frac{68}{27}$. The requested sum is $\boxed{\textbf{(C) } 95}$.

Problem 21:

Arby the ant is at $(0, 0)$ facing in the positive x direction. On the first day he walks one unit to the point $(1, 0)$. Each day after that, he randomly turns either 60° left or 60° right, and walks one third the distance he travelled the previous day. What is Arby's expected x-coordinate as time goes to infinity?

(A) $\frac{6}{5}$ **(B)** $\frac{9}{7}$ **(C)** $\frac{4}{3}$ **(D)** $\frac{3}{2}$ **(E)** $\frac{8}{5}$

Proposed by ihatemath123.

Solution 1: Let E be Arby's expected x-coordinate as time goes to infinity. After he crawls to $(1, 0)$.

Now, he starts walking a third of the distance he travelled the previous day. Suppose Arby turns 60° left. By definition of E and the fact that all future movements are a $\frac{1}{3}$ of the previous move, it is expected Arby will walk a distance of $\frac{1}{3}E$ in the direction 60° left of the positive x direction, as time goes to infinity. With the 60° angle, this is equivalent to $\frac{1}{2} \cdot \frac{1}{3}E$ in the x direction. A similar argument holds for when he turns 60° right in the x direction for his first turn, instead.

So his expected movement in the x direction from $(1, 0)$ is $\frac{E}{6}$. Thus, the expected x-coordinate as time goes to infinity is $1 + \frac{E}{6}$. But by definition, E is the expected x-coordinate, so

$$1 + \frac{E}{6} = E \implies E = \boxed{\textbf{(A) } \frac{6}{5}}.$$

Solution 2 (due to Brainfertilzer)[2]*:* Toss Arby onto the complex plane, starting out at 0, pointing in the positive real direction. Let z be the complex number representing Arby's displacement each day, starting at 1. After the first day, z is multiplied by either $\frac{\text{cis}(120^\circ)}{3}$ or $\frac{\text{cis}(240^\circ)}{3}$, chosen at random.

Because the problem asks us to find Arby's *expected* position, and because his 50-50 decision each day is made independently of other decisions, we may apply linearity of expectation. On average, z is multiplied by

$$\frac{1}{2}\left(\frac{\text{cis}(120^\circ)}{3} + \frac{\text{cis}(240^\circ)}{3}\right) = \frac{1}{6}$$

at the end of each day.

So, Arby's average final position is just

$$1 + \frac{1}{6} + \frac{1}{36} + \cdots = \boxed{\textbf{(A) } \frac{6}{5}}.$$

[2] Oops, when I wrote this problem I didn't realize that this solution existed! It makes the first solution kind of redundant. -ihatemath123

Problem 22:

The fourth largest divisor of an integer is 165. What is the sum of all possible values of its fifth largest divisor?

(A) 237 **(B)** 289 **(C)** 325 **(D)** 493 **(E)** 511

Proposed by ihatemath123.

Let $1 = d_1 < d_2 < ... < d_k = n$ be the k divisors of n. A property of the divisors is that the product of the mth smallest divisor and mth largest divisor is n for all m with $1 \le m \le k$.

Thus, $n = 165d_4$. Clearly, $d_4 \ge 4$. Since $165 = 3 \cdot 5 \cdot 11$, it follows that n is necessarily divisible by 1, 3, 5, and 11. Thus, $d_4 \le 11$.

- If $d_4 = 4$, then $1, 2, 3, 4$ are all divisors of $n = 165 \cdot 4$, meaning d_4 is indeed the 4th smallest divisor of n.
- If $d_4 > 5$ and is even, then 2 is necessarily a divisor of n. Then, 1, 2, 3, and 5 are all less than d_4 and are necessarily divisors of n. This makes it impossible for d_4 to actually be the 4th smallest divisor of n.
- If $d_4 = 5$, then 2 and 4 do not divide $n = 165 \cdot 5$. This means that d_4 is the 3rd smallest divisor of n, not the 4th smallest.
- If $d_4 \in \{7, 9, 11\}$, then $n = 165d_4$ has no even divisors. Furthermore, n doesn't have any odd divisors less than d_4, besides 1, 3, and 5. Thus, d_4 is indeed the 4th smallest divisor of n.

Thus, the only possible values of d_4 are 4, 7, 9, and 11.

Since d_{k-4} is the 5th largest divisor of n, it follows that $d_5 d_{k-4} = n = 165d_4$. This implies $d_{k-4} = \frac{165d_4}{d_5}$.

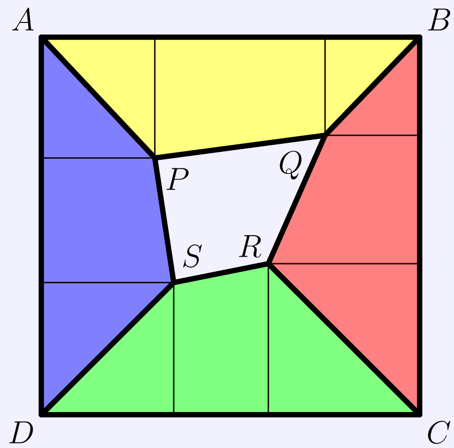
- If $d_4 = 4$, then $d_5 = 5$, since 5 divides $165 \cdot 4$. Hence, $d_{k-4} = \frac{165 \cdot 4}{5} = 132$.
- If $d_4 = 7$, then $d_5 = 11$, since 11 divides $165 \cdot 7$, but 9 and even numbers do not. Hence, $d_{k-4} = \frac{165 \cdot 7}{11} = 105$.
- If $d_4 = 9$, then $d_5 = 11$, since 11 divides $165 \cdot 7$, but 10 (an even number) does not. Hence, $d_{k-4} = \frac{165 \cdot 9}{11} = 135$.
- If $d_4 = 11$, then $d_5 = 15$, since 15 divides $165 \cdot 7$, but 13 and even numbers do not. Hence, $d_{k-4} = \frac{165 \cdot 11}{15} = 121$.

The sum of all possible values of d_{k-4} is

$$132 + 105 + 135 + 121 = \boxed{\textbf{(D) } 493}.$$

Problem 23:

Four squares are placed in each corner of a larger square, as shown in the diagram below. If the areas of quadrilaterals $APQB$, $BQRC$, $CRSD$ and $DSPA$ are 20, 23, 26 and 25, respectively, what is the area of quadrilateral $PQRS$?



(A) 13 **(B)** 14 **(C)** 15 **(D)** 16 **(E)** 17

Proposed by kante314.

Let O be the center of square $ABCD$. Note that P, O, and R all lie on diagonal $\overline{AC}$ and Q, O, and S all lie on diagonal $\overline{BD}$. Let

$$K = [AOB] = [BOC] = [COD] = [DOA] = \frac{[ABCD]}{4}.$$

Then, $[POQ] = [AOB] - [APQB] = K - 20$. Similarly,

$$[QOR] = K - 23, \quad [ROS] = K - 26, \quad [SOP] = K - 25.$$

Diagonals AC and BD are perpendicular in the square, so PR and QS are perpendicular, too. Since PR and QS intersect at R,

$$[POQ] \cdot [ROS] = \frac{OP \cdot OQ}{2} \cdot \frac{OR \cdot OS}{2} = \frac{OP \cdot OQ \cdot OR \cdot OS}{4}.$$

Similarly,

$$[QOR] \cdot [SOP] = \frac{OQ \cdot OR}{2} \cdot \frac{OS \cdot OP}{2} = \frac{OP \cdot OQ \cdot OR \cdot OS}{4}.$$

Hence, $[POQ] \cdot [ROS] = [QOR] \cdot [SOP]$, implying

$$(K - 20)(K - 26) = (K - 23)(K - 25) \implies K^2 - 46K + 520 = K^2 - 48K + 575 \implies K = \frac{55}{2}.$$

Hence, $[ABCD] = 4K = 110$, so

$$[PQRS] = [ABCD] - [APQB] - [BQRC] - [CRSD] - [DSPA] = \boxed{\textbf{(D) } 16}.$$

Problem 24:

How many polynomials $P(x)$ with nonnegative integer coefficients are there such that

$$P(x) \leq 2x^3 + 2x^2 + 2$$

for all real numbers x?

(A) 15 **(B)** 16 **(C)** 17 **(D)** 18 **(E)** 19

Proposed by ihatemath123

Define $Q(x) = 2x^3 + 2x^2 + 2$ and $R(x) = P(x) - Q(x)$. Thus, $R(x) \leq 0$ for all real numbers x.

Claim: $R(x)$ must have a degree of 0 or 2.

Since $R(x) \leq 0$, the leading coefficient of $R(x)$ must be negative and its degree must be even. If $R(x)$ contains a term with a degree greater than 3, this term must have come from $P(x)$ (seeing as $Q(x)$ is degree 3). But since $P(x)$ has nonnegative integer coefficients, this is a contradiction, so the degree of the polynomial is less than or equal to 3. Since it must also be even, the degree has to be 0 or 2. $\square$

Consider cases on the degree of $R(x) = (c_2 - 2)x^2 + c_1 x + (c_0 - 2)$, which can only be 0 or 2 by the claim.

Case 1: $R(x)$ is of degree 0

Then, $c_2 = 2$ and $c_1 = 0$. Then $c_0 - 2 = R(x) \leq 0$ for all real x. With $c_0 \geq 0$, there are 3 possible values of c_0, namely those in $[0, 2]$. This case yields 3 polynomials.

Case 2: $R(x)$ is of degree 2

Then, $c_2 \neq 2$ with $R(x) \leq 0$.

For a quadratic polynomial to be less than or equal to 0 for all values of x, it must point downwards. Otherwise, it will eventually be positive for some x. Furthermore, the quadratic cannot have 2 distinct roots. Otherwise, the quadratic is positive for values of x between those roots.

Hence, $c_2 - 2 < 0$ and the discriminant of R is non-positive, implying $c_1^2 - 4(c_2 - 2)(c_0 - 2) \leq 0$. Since c_2 is nonnegative, c_2 is either 0 or 1.

- If $c_2 = 0$, then $c_1^2 \leq 8(2 - c_0)$.
  - If $c_0 = 0$, then $0 \leq c_1 \leq 4$, yielding 5 polynomials.
  - If $c_0 = 1$, then $0 \leq c_1 \leq 2$, yielding 3 polynomials.
  - If $c_0 = 2$, then $c_1 = 0$, yielding 1 polynomial.
  - Other nonnegative values of c_0 force the RHS of the inequality to be negative and the LHS to be positive.
- If $c_2 = 1$, then $c_1^2 \leq 4(2 - c_0)$.
  - If $c_0 = 0$, then $0 \leq c_1 \leq 2$, yielding 3 polynomials.
  - If $c_0 = 1$, then $0 \leq c_1 \leq 2$, yielding 3 polynomials.
  - If $c_0 = 2$, then $c_1 = 0$, yielding 1 polynomial.
  - Other nonnegative values of c_0 force the RHS of the inequality to be negative and the LHS to be positive.

The total for Case 2 is $5 + 3 + 1 + 3 + 3 + 1 = 16$.

Thus, the number of polynomials in total is $3 + 16 = \boxed{\textbf{(E) } 19}$.

Problem 25:

How many nonempty subsets of $\{1, 2, \ldots, 12\}$ have an integer median?

(A) 2730 **(B)** 2731 **(C)** 3070 **(D)** 3071 **(E)** 3072

Proposed by ihatemath123

The 2^{11} subsets with an odd number of elements clearly have an integer median, so hereon we consider only subsets with an even number of elements.

Claim: There is a one-to-one correspondence between even-sized subsets with an integer median, and even-sized subsets whose smallest element is even.

Starting from an even-sized subset with an integer median, let $x < y$ be the two middle numbers. So, $y - x$ is even. Now, we add 0 to our subset, decrease every element by x and take negative values mod 13. Since we decreased x by x, there will be a 0 in our new subset which we now delete. Now, y is replaced with $y - x$, an even value which much be the smallest element in our new subset.

If we begin from an even-sized subset whose smallest element is even, a similar process works: let $x < y$ be the two middle numbers, add 0 to the subset, decrease every element by y, take negative values mod 13, delete 0 and we are left with an even-sized subset with an integer median. $\square$

If the smallest element of a subset is i, there are 2^{12-i} ways to choose the remaining subset. Exactly half of these yield a subset with an even number of elements; there are 2^{11-i} ways to choose the remaining subset such that the subset has an even number of elements. So, our final answer is

$$2^{11} + \left(2^9 + 2^7 + 2^5 + 2^3 + 2^1\right) = \boxed{\textbf{(A) } 2730}.$$