ZeMC 12 Results and Solutions

Ze Committee

Thursday, November 24th

This document contains results & solutions for the ZeMC 12, a mock AMC 12 held for the 2022-2023 competition season. The test can be found here. If you have any questions, you should contact ihatemath123 on AoPS, or imagine dragon#3311 on Discord. You can also check the ZeMC public discussion forum.

(Disclaimer from ihatemath123: I couldn't get the diagrams working in the problem statements, so the statements are diagramless.)

The ZeMC competition series is made possible by the contributions of the following problem-writers and test-solvers:

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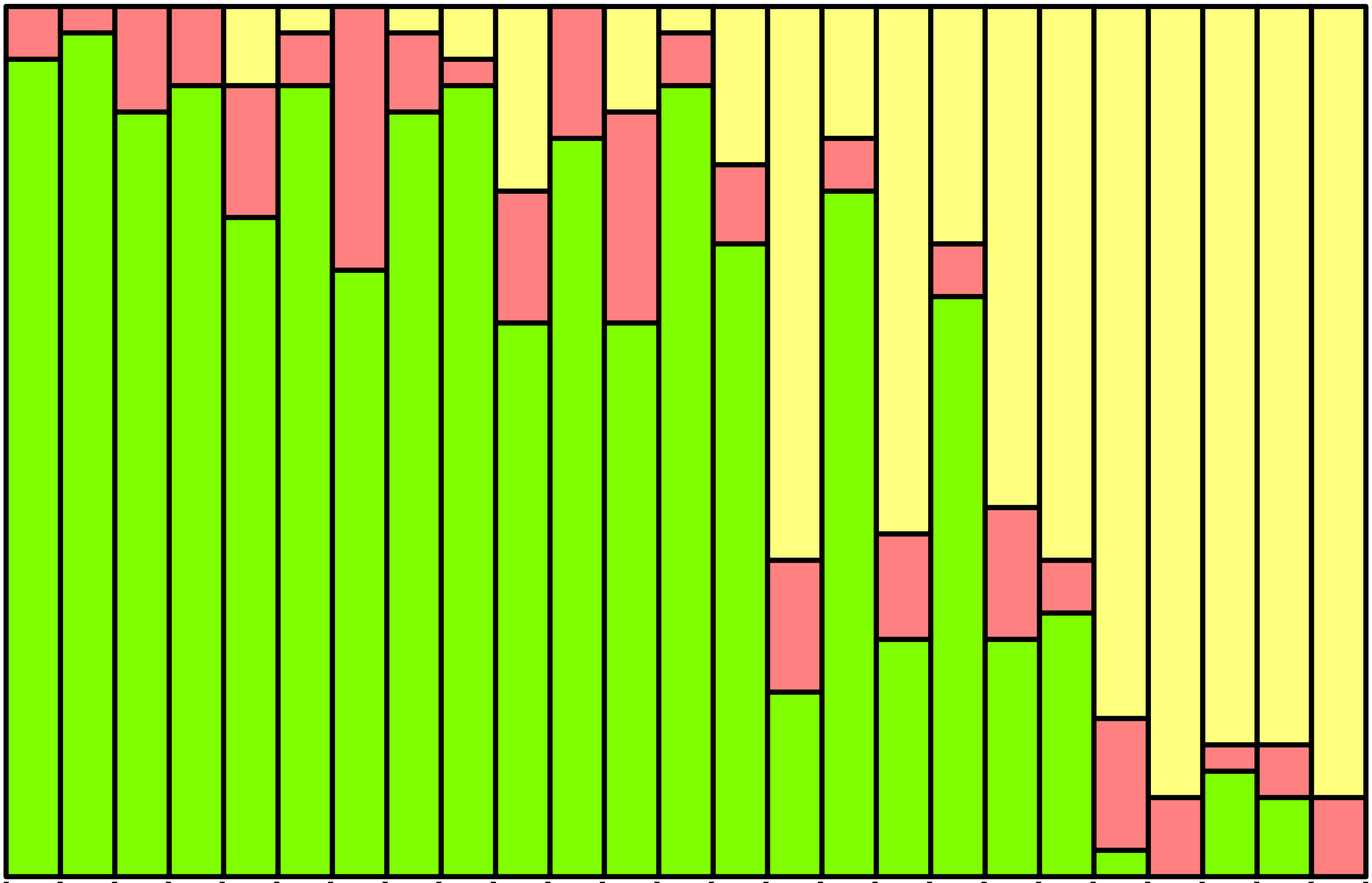
This solutions manual was written by

P_Groudon.

Answer Key and Solve Rates

The chart of solve rates below reflects the number of correct, incorrect and blank answers given for each problem on the contest. In the chart, green denotes a correct answer, red denotes an incorrect answer and yellow denotes no answer.



1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
B	B	B	E	B	E	B	E	E	B	D	A	E	D	E	A	A	A	D	E	E	E	D	D	C

Problem 1:

What is $1 \times 1.5 \times 2 \times 2.5 \times 3 \times 3.5 \times 4$?

(A) 180 **(B)** 315 **(C)** 360 **(D)** 630 **(E)** 720

Proposed by ihatemath123

Compute as follows:

$$1 \times 1.5 \times 2 \times 2.5 \times 3 \times 3.5 \times 4 = \frac{3}{2} \times 2 \times \frac{5}{2} \times 3 \times \frac{7}{2} \times 4 = 3 \times 5 \times 3 \times 7 = \boxed{\textbf{(B) } 315}.$$

Problem 2:

Mira lines the perimeter of a rectangle with red and blue string, such that 3 sides are lined with red string, and 1 side is lined with blue string. If she uses 11 inches of red string and 6 inches of blue string, what is the area of Mira's rectangle?

(A) 9 **(B)** 15 **(C)** 18 **(D)** 21 **(E)** 30

Proposed by ihatemath123

Let x and y be the dimensions of the rectangle with x being the dimension lined with the blue string. Then, by the given information, $x = 6$ and $x + 2y = 11$. Solving these equations, $y = \frac{5}{2}$. Thus, $xy = \boxed{\textbf{(B) } 15}$.

Problem 3:

Ms. Debarlaben must buy at least 7 boxes of pencils to supply at least one pencil to each student in her class. However, she accidentally buys only 4 boxes of pencils, providing her with 10 fewer pencils than students. How many students are in Ms. Debarlaben's class? (Each box has the same number of pencils.)

(A) 24 **(B)** 26 **(C)** 28 **(D)** 30 **(E)** 32

Proposed by ihatemath123

Suppose each box contains p pencils and there are s students in the class. Since at least 7 boxes of pencils are necessary, this implies that 6 boxes of pencils will be too few. Thus, $6p < s \leq 7p$. Furthermore, it is given that $4p + 10 = s$. It follows that

$$6p < 4p + 10 \leq 7p \implies 2p < 10 \leq 3p \implies p = 4.$$

Thus, $s = 4 \cdot 4 + 10 = \boxed{\textbf{(B) } 26}$.

Problem 4:

A trapezoid has area $\mathcal{A}$. If one of its base lengths were halved, its area would be $\frac{2}{3}\mathcal{A}$. If *instead* the other base length were halved, what would the trapezoid's area be?

(A) $\dfrac{1}{3}\mathcal{A}$ **(B)** $\dfrac{1}{2}\mathcal{A}$ **(C)** $\dfrac{2}{3}\mathcal{A}$ **(D)** $\dfrac{3}{4}\mathcal{A}$ **(E)** $\dfrac{5}{6}\mathcal{A}$

Proposed by bissue

Let the bases of the trapezoid be x and y and the height be h. Suppose x is the base halved first. Then, by the given information,

$$\frac{x+y}{2}\cdot h = \mathcal{A}, \qquad \frac{\frac{x}{2}+y}{2}\cdot h = \frac{2}{3}\mathcal{A}$$
$$\implies x+y = \frac{2\mathcal{A}}{h}, \qquad \frac{x}{2}+y = \frac{4\mathcal{A}}{3h}.$$

Solving for x and y in terms of $\mathcal{A}$ and h implies $(x,y) = (\frac{4\mathcal{A}}{3h}, \frac{2\mathcal{A}}{3h})$. Thus, if y were the base that is halved, the area of the trapezoid would be,

$$\frac{x+\frac{y}{2}}{2}\cdot h = \boxed{\textbf{(E) } \frac{5}{6}\mathcal{A}}.$$

Problem 5:

Albert writes down all of the even integers between 2 and 2048, inclusive. What is the sum of the three digits that he writes down the most number of times?

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

Proposed by ihatemath123

Due to the thousands digit, 1 appears in roughly half the numbers, far greater than any other digit.

Next, 2 appears a similar amount of times to the digits $3, 4, 5, \dots, 9$; however, it appears as the thousands digit quite a few times, and in the tens digit too; these advantages make 2 better than $3, 4, 5, \dots, 9$.

Similarly, 4 is just 2 but never appears in the thousands digit. However, it's better than 3 due to it appearing in the units digit more, and better than $5, 6, 7, 8, 9$ due to appearing in the tens digit more.

Now, just note that 0 doesn't appear very much; it never appears as a leading digit, so for all the 1's we got from $100, 102, 104, \dots, 198$, there was no corresponding $000, 002, 004, \dots, 098$.

We have that $1+2+4 = \boxed{\textbf{(B) } 7}$.

Problem 6:

Trains A and B simultaneously depart from Detroit and Chicago, respectively, and travel at constant but different rates towards the other city, along the same route. The trains meet exactly 3.5 hours after their departure; 2.5 hours after the meetup, train A arrives in Chicago. How many minutes after the meetup does train B arrive in Detroit?

(A) 270 **(B)** 276 **(C)** 282 **(D)** 288 **(E)** 294

Proposed by ihatemath123

Let the meeting point of the trains be X. It takes train A 2.5 hours to travel from X to Chicago, and it takes train B 3.5 hours to travel from Chicago to X. Because the trains travel at constant rates, for any distance, train B will take $\frac{3.5 \text{ hrs}}{2.5 \text{ hrs}} = 1.4$ times as long to go that distance as train A. It takes train A 3.5 hours to travel from Detroit to X, so it will take train B

$$3.5 \text{ hrs} \cdot 1.4 = \boxed{\textbf{(E) } 294}$$

minutes to travel from X to Detroit.

Problem 7:

What is the smallest positive integer n for which precisely 6 of the integers n, $2n$, $3n$, $\ldots$, $1000n$ are perfect squares?

(A) 20 **(B)** 21 **(C)** 26 **(D)** 27 **(E)** 28

Proposed by ihatemath123

The numbers

$$n \cdot (n), n \cdot (4n), n \cdot (9n), n \cdot (16n), \ldots$$

are always perfect squares, regardless of n. Thus, if $n \cdot (49n) \leq 1000n$, there will be at least 7 integers that are perfect squares. Thus, $n \cdot (49n) > 1000n \implies n > 20.41$. Since 21 is square free, the *only* terms $n, 2n, 3n, \ldots, 1000n$ that are perfect squares are those in the form $21k^2 \cdot n$ for some positive integer k or $21n, 84n, 189n, ..., 756n$.

We have shown a lower bound of 21 and found that it is attainable, so our answer is $\boxed{\textbf{(B) } 21}$.

Problem 8:

There exists a unique integer k for which the points $(2k, 22)$, $(11, 4k)$, and $(2022, -2026)$ lie on a line in the coordinate plane. What is the sum of the digits of k?

(A) 18 **(B)** 19 **(C)** 20 **(D)** 21 **(E)** 22

Proposed by kante314 and bissue

Three points A, B, and C being collinear is equivalent to the slopes of lines AB and AC being equal. If $A = (2k, 22)$, $B = (11, 4k)$, and $C = (2022, -2026)$, the slope of line AB is $\frac{4k-22}{11-2k} = -2$. The slope of line AC is $\frac{-2026-22}{2022-2k} = -\frac{1024}{1011-k}$. Setting these two equal,

$$-2 = -\frac{1024}{1011-k} \implies 1011 - k = 512 \implies k = 499.$$

The sum of the digits is $\boxed{\textbf{(E) } 22}$.

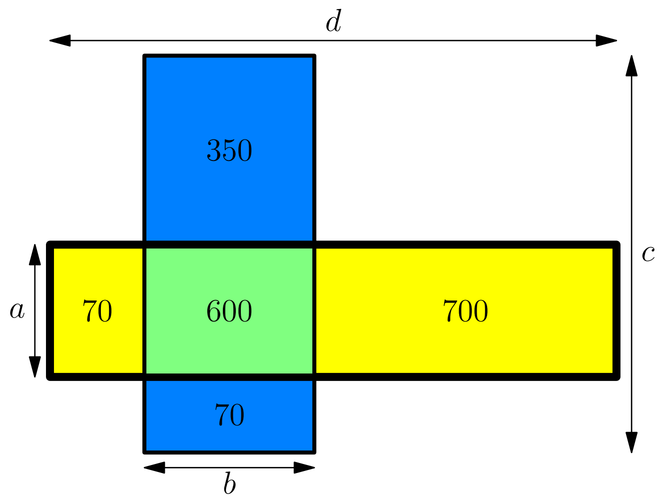
Problem 9:

Two rectangles with integer side lengths are drawn on a plane, forming 5 smaller rectangles, whose areas are indicated in the diagram below. What is the sum of the perimeters of these two rectangles?

(A) 440 **(B)** 442 **(C)** 444 **(D)** 446 **(E)** 448

Proposed by ihatemath123

Labels of the dimensions are shown below and by the given information, each variable is an integer:



Using area, $ab = 600$, $ad = 70 + 600 + 700 = 1370$, and $bc = 70 + 600 + 350 = 1020$.

Note that $1370 = 137 \cdot 10$ and 137 is prime. Since 600 does not have a factor of 137, a cannot contain a factor of 137 by the first equation. Thus, by the second equation, a is a divisor of 10.

Similarly, note that $1020 = 17 \cdot 60$ and 17 is prime. Since 600 does not have a factor of 17, b cannot contain a factor of 17 by the first equation. Thus, by the third equation, b is a divisor of 60.

With these two conditions, the only way $ab = 600$ is satisfies is if $a = 10$ and $b = 60$ because otherwise, ab would obviously be too small. Then, $c = 17$ and $d = 137$. Thus, the sum of the perimeters of the rectangle is $2(10 + 137 + 60 + 17) = \boxed{\textbf{(E) } 448}$.

Problem 10:

The angle bisectors of $\triangle ABC$ all pass through a single point, I. If the coordinates of points I, A and B are $(1, 2)$, $(4, 2)$ and $(-2, -1)$, respectively, find the sum of the coordinates of point C.

(A) 4 **(B)** $\dfrac{21}{5}$ **(C)** $\dfrac{13}{3}$ **(D)** $\dfrac{9}{2}$ **(E)** $\dfrac{29}{6}$

Proposed by ihatemath123

We are given that BI bisects $\angle ABC$ and AI bisects $\angle BAC$. Thus, line BC is obtained by reflecting AB about BI.

Line AB has slope $\frac{1}{2}$. Since line BI has slope 1, the slope of line BC will be the reciprocal of that of AB or 2. Thus, the equation for line BC is $y + 1 = 2(x + 2)$. Similarly, line AC is obtained by reflecting AB about AI. Since line AI has a slope of 0, the slope of line AC will be the negative of that of AB or $-\frac{1}{2}$. Thus, the equation for line AC is $y - 2 = -\frac{1}{2}(x - 4)$. Solving these equations for their intersection point, C, implies $(x, y) = (\frac{2}{5}, \frac{19}{5})$. The requested sum is $\boxed{\textbf{(B) } \dfrac{21}{5}}$.

Problem 11:

A town has exactly 100 bakers, making up n percent of its population, for a positive integer n between 1 and 100 inclusive. How many possible values of n are there?

(A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

Proposed by ihatemath123

These 100 bakers make up $\frac{n}{100}$ of the population, so multiplying the number of bakers by 100 and dividing by n yields the total population. Thus, $\frac{10{,}000}{n}$ must be an integer.

Without the $1 \le n \le 100$ bound, n could be each of the 25 divisors of $10{,}000$. With the exception of 100, for any integer $n = k$ that does not fall in the bound, $n = \frac{10{,}000}{k}$ falls in the bound. However, 100 falls into the bound. Therefore, besides 100, exactly half of the divisors fall into the bound, but 100 must be added back. The answer is

$$\frac{25-1}{2} + 1 = \boxed{\textbf{(D) } 13}.$$

Problem 12:

How many ways can each face of a cube be colored one of four colors, so that no two faces that share an edge are the same color? (Rotations and reflections of arrangements are considered distinct from each other.)

(A) 96 **(B)** 120 **(C)** 144 **(D)** 168 **(E)** 192

Proposed by ihatemath123

Label the faces of the cube so that A and B are opposite faces. Then, label the four remaining faces C, D, E, and F in that order around the cube.

Case 1: A and B are the same color

There are 4 ways to choose the color for these two faces.

- If C and E are the same color, there are 3 ways to choose this color. Then, D and F can be either of the 2 colors not used yet without restriction, leading to 2^2 ways to color these faces.
- If C and E are not the same color, there are 3 ways to choose the color for C and then 2 ways to choose the color for E. Then, D and F must both be the last color not used yet.

Thus, the total number of colorings in this case is $4(3 \cdot 2^2 + 3 \cdot 2) = 72$.

Case 2: A and B are different colors

There are 4 ways to choose the color for A, 3 ways to choose the color for B, and then 2 ways to choose the color for C. D and F are forced to be the last color not used yet. This in turn forces E to be the same color as C.

Thus, the total number of colorings in this case is $4 \cdot 3 \cdot 2 = 24$.

Combining the two cases, the total is $72 + 24 = \boxed{\textbf{(A) } 96}$.

Problem 13:

A rectangular prism has a volume of 15 and a surface area of 40. When each of its edge lengths are increased by 3, its volume becomes 150, and its surface area becomes x. What is x?

(A) 150 **(B)** 152 **(C)** 154 **(D)** 156 **(E)** 158

Proposed by ihatemath123

Let the dimensions of the prism be a, b, and c. The given information implies $abc = 15$ and $2(ab+bc+ca) = 40$ or $ab + bc + ca = 20$. Since the volume of the new prism is 150,

$$\begin{aligned} (a+3)(b+3)(c+3) &= 150 \\ abc + 9(a+b+c) + 3(ab+bc+ca) + 27 &= 150 \\ 15 + 9(a+b+c) + 3 \cdot 20 &= 123 \\ a+b+c &= \frac{16}{3}. \end{aligned}$$

Thus, the surface area of the new prism is

$$\begin{aligned} 2((a+3)(b+3)+(b+3)(c+3)+(c+3)(a+3)) &= 2(3 \cdot 9 + 6(a+b+c) + (ab+bc+ca)) \\ &= 2\left(27 + 6 \cdot \frac{16}{3} + 20\right) = \boxed{\textbf{(E) } 158}. \end{aligned}$$

Problem 14:

Zoe and Kylie each choose six distinct single digit positive integers and take the product of their numbers. If Zoe's product is 28 greater than Kylie's product, what is the sum of Zoe's six numbers?[a]

(A) 25 **(B)** 26 **(C)** 27 **(D)** 28 **(E)** 29

Proposed by ihatemath123

[a]With a computer program, one can verify that 28 is the smallest nonzero difference that Zoe and Kylie could obtain.

The difference between the two products is 28, which is not divisible by 3. Thus, Zoe and Kylie cannot both have a multiple of 3 in their set of six digits.

If neither person has a multiple of 3 in their set of six digits, then their set of digits will be the same. This implies a difference of 0 in their products, but this is clearly a contradiction.

Hence, exactly one of the two people has no multiple of 3 in their set of digits.

It follows that $3, 6$, and 9 has been ruled out from one of two girls' sets, so either Zoe or Kylie has the set $\{1, 2, 4, 5, 7, 8\}$. The product of these digits is $28 \cdot 80$. Since the two products have a difference of 28, the other product is either $28 \cdot 79$ or $28 \cdot 81$. The former is impossible, since 79 is prime, so no single digit positive integer will contain a factor of 79.

Thus, the other product is $28 \cdot 81$. Since this is the bigger product, it is Zoe's. In addition, the product of the digits that Zoe does not use will be $\frac{9!}{28 \cdot 81} = 160$, which forces the unused digits to be $\{4, 5, 8\}$. Thus, Zoe's digits are $\{1, 2, 3, 6, 7, 9\}$, which have a sum of $\boxed{\textbf{(D) } 28}$. [1]

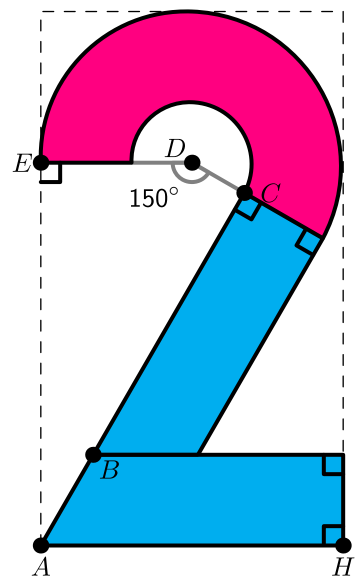
[1]Coincidentally, the answer is the same as the given number in the problem!

Problem 15:

A "2" figure is constructed by placing together two congruent trapezoids and one annulus sector (the region bounded by two concentric circles and two radii of the larger circle), as shown in the diagram below. The figure can be inscribed in a rectangle such that the annulus sector touches the boundary of the rectangle three times. The ratio of the height to the width of the rectangle can be written as $\frac{a+\sqrt{b}}{c}$ for positive integers a, b and c, where $\gcd(a,b,c)=1$. What is $a+b+c$?

(A) 90 **(B)** 91 **(C)** 92 **(D)** 93 **(E)** 94

Proposed by ihatemath123



Concise explanation: Define $f(\vec{v})$ as the x-component of $\vec{v}$.
Since $\overrightarrow{ED}+\overrightarrow{DC}+\overrightarrow{CB}+\overrightarrow{BA}=\overrightarrow{EA}$, it follows that

$$f\left(\overrightarrow{ED}\right)+f\left(\overrightarrow{DC}\right)+f\left(\overrightarrow{CB}\right)+f\left(\overrightarrow{BA}\right)=f\left(\overrightarrow{EA}\right)=0.$$

Furthermore, observe that $CB=AH=2\times ED$. The argument of $\overrightarrow{CB}$ is $240°$, so its x-component is equal to $-\frac{\left|\overrightarrow{CB}\right|}{2}$. Therefore,

$$f\left(\overrightarrow{ED}\right)+f\left(\overrightarrow{CB}\right)=0.$$

Subtracting this from our previous equation, we obtain that

$$f\left(\overrightarrow{DC}\right)+f\left(\overrightarrow{BA}\right)=0.$$

Due to the arguments of $\overrightarrow{DC}$ and $\overrightarrow{BA}$, this means that $\left|\overrightarrow{DC}\right|$ and $\left|\overrightarrow{BA}\right|$ are the legs lengths of a 30-60-90 triangle; that is,

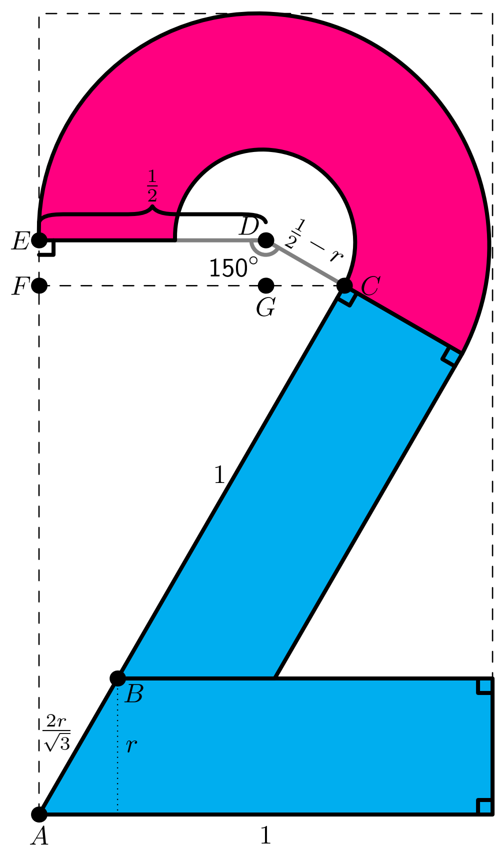
$$\frac{BA}{DC}=\sqrt{3}.$$

Also due to 30-60-90 ratios, the length BA is $\frac{2}{\sqrt{3}}$ times the constant "thickness" of this two (i.e. The distance from B to $\overline{AH}$, also equivalent to the thickness of the annulus sector). So,

$$\begin{aligned}\frac{BA}{DC} &= \sqrt{3}\\ \frac{\text{Thickness of two}\cdot\frac{2}{\sqrt{3}}}{DC} &= \sqrt{3}\\ \frac{\text{Thickness of two}}{DC} &= \frac{3}{2}.\end{aligned}$$

Now, we can set $ED = 1$ without loss of generality, and compute $\overrightarrow{EA}$; adding this to the length ED will give us the height of the rectangle. If we set $ED = 1$, the width of the rectangle is clearly 2. This gives us a final answer of $\boxed{\textbf{(E) } 94}$. [2]

Thorough explanation: Without loss of generality, we may assume that the width of this rectangle is 1. Let the thickness of the shape overall be r. Then, we may label the following side lengths, and name the following vertices:



(We define F as the perpendicular from C to $\overline{AE}$ and G as the perpendicular from D to $\overline{CF}$.)

Note that the acute angle in each trapezoid has a measure of $60°$. We will find the length CF in two different

[2]This problem is not as bashy as it looks, I promise :) - ihaetmath123

ways, both in terms of r. On one hand, we have that

$$\begin{aligned} CF &= CG + GF \\ &= \frac{\sqrt{3}}{2} CD + DE \\ &= \frac{\sqrt{3}}{2}\left(\frac{1}{2} - r\right) + \frac{1}{2} \\ &= \frac{2+\sqrt{3}}{4} - \frac{r\sqrt{3}}{2}. \end{aligned}$$

On the other hand,

$$\begin{aligned} CF &= \frac{CA}{2} \\ &= \frac{CB}{2} + \frac{BA}{2} \\ &= \frac{1}{2} + \frac{r}{\sqrt{3}}. \end{aligned}$$

Setting these expressions equal, we have that

$$\begin{aligned} \frac{1}{2} + \frac{r}{\sqrt{3}} &= \frac{2+\sqrt{3}}{4} - \frac{r\sqrt{3}}{2} \\ r\left(\frac{1}{\sqrt{3}} + \frac{\sqrt{3}}{2}\right) &= \frac{\sqrt{3}}{4} \\ r &= \frac{3}{10}. \end{aligned}$$

We compute the height of the rectangle to be

$$\begin{aligned} & AE + \frac{1}{2} \\ = \;& AF + FE + \frac{1}{2} \\ = \;& \frac{AC\sqrt{3}}{2} + \frac{CD}{2} + \frac{1}{2} \\ = \;& \frac{3+5\sqrt{3}}{10} + \frac{1}{10} + \frac{1}{2} \\ = \;& \frac{9+5\sqrt{3}}{10}. \end{aligned}$$

Since we assigned the width as 1, the height is just the ratio, so by extraction, our answer is $\boxed{\textbf{(E) } 94}$.

Problem 16:

The polynomial $x^3 + ax^2 + bx + c$ has exactly two distinct real roots. If these two roots are equal to a and b, what is the sum of all possible values of c?

(A) $-\dfrac{44}{125}$ **(B)** $-\dfrac{11}{64}$ **(C)** $-\dfrac{3}{8}$ **(D)** $\dfrac{95}{216}$ **(E)** $\dfrac{16}{27}$

Proposed by asbodke

Since the cubic has 3 roots (not necessarily distinct) but 2 distinct roots, one root appears twice while the other appears once.

Case 1: a appears twice and b appears once

Then, the polynomial is in the form

$$(x-a)^2(x-b) = x^3 + (-2a-b)x^2 + (a^2+2ab)x - a^2b.$$

Comparing coefficients yields the following system:

$$\begin{aligned} a &= -2a - b \\ b &= a^2 + 2ab \\ c &= -a^2b. \end{aligned}$$

The first equation implies $b = -3a$. Plugging this into the second equation implies

$$-3a = a^2 - 6a^2 \implies 5a^2 = 3a \implies a \in \left\{0, \frac{3}{5}\right\}.$$

If $a = 0$, then $c = 0$ by the third equation, which does not affect the requested quantity. If $a = \frac{3}{5}$, then $b = -\frac{9}{5}$ from the first equation. Then, the third equation implies $c = \frac{81}{125}$.

Case 2: b appears twice and a appears once

Similar to Case 1,

$$(x-a)(x-b)^2 = x^3 + (-a-2b)x^2 + (b^2+2ab)x - ab^2.$$

Comparing coefficients yields the following system:

$$\begin{aligned} a &= -a - 2b \\ b &= b^2 + 2ab \\ c &= -ab^2. \end{aligned}$$

The first equation implies $b = -a$. Plugging this into the second equation implies

$$-a = a^2 - 2a^2 \implies a^2 = a \implies \{0, 1\}.$$

Like before, $a = 0$ implies $c = 0$, which does not affect the requested quantity. If $a = 1$, then $b = -1$ from the first equation. Then, the third equation implies $c = -1$.

Thus, the sum of all possible values of c is $\frac{81}{125} + (-1) = \boxed{\textbf{(A)}\ -\frac{44}{125}}$.

Problem 17:

Eight points lie on a plane such that:

- No two points are less than 4 units apart, and
- The smallest convex polygon containing all eight points is a regular hexagon with side length s.

The minimum possible value of s can be written as $\sqrt{m}+\sqrt{n}$ for integers m and n. What is $m+n$? (Here, a point is *contained* in a polygon if it lies on or inside the boundary of the polygon.)

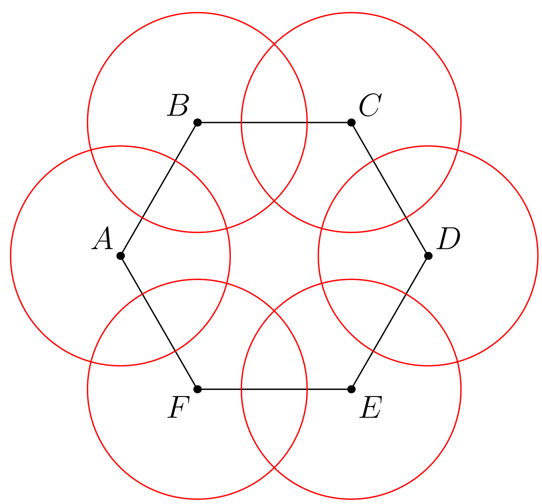
(A) 18 **(B)** 20 **(C)** 22 **(D)** 24 **(E)** 26

Proposed by asbodke

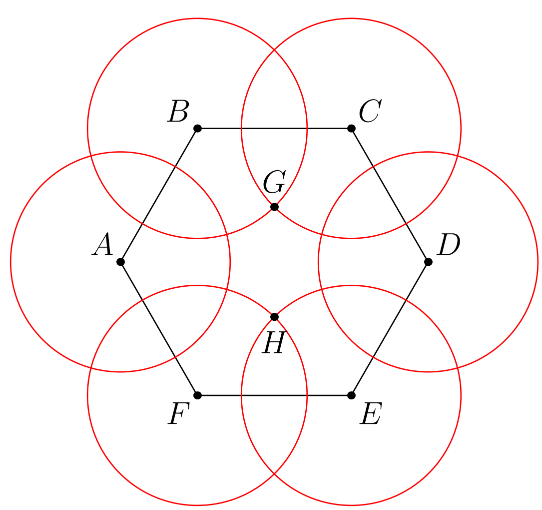
For the smallest polygon containing the eight points to specifically be a regular hexagon, 6 of the points must be the vertices of a regular hexagon and the other 2 points must be inside the hexagon. Any other arrangement of the outer points will result in a polygon of a different shape containing the eight points.

Let $ABCDEF$ be the regular hexagon with side length s. Clearly, $s \geq 4$, since the points must be at least 4 units apart.

For each vertex of the hexagon, draw a circle of radius 4 with its center at the vertex:



The remaining two points inside the hexagon cannot be inside any of the circles. Otherwise, two points will be less than 4 units apart. Thus, for s to be minimal, the hexagon needs to be just big enough that two points G and H that are 4 units apart can be placed as shown, where G and H lie on the intersection of two circles:



To compute s, let M and N be the midpoints of $\overline{BC}$ and $\overline{EF}$, respectively, so that M, G, H, and N are collinear. Since $BG = 4$ and $BM = \frac{s}{2}$. By Pythagorean Theorem, $MG = \sqrt{16 - \frac{s^2}{4}}$. By symmetry, $HN = MG$. Since $GH = 4$,

$$MN = MG + GH + HN = 4 + 2\sqrt{16 - \frac{s^2}{4}}.$$

By the properties of a regular hexagon, $MN = BF = s\sqrt{3}$. Thus,

$$s\sqrt{3} - 4 = 2\sqrt{16 - \frac{s^2}{4}} \implies 3s^2 - 8s\sqrt{3} + 16 = 4\left(16 - \frac{s^2}{4}\right) \implies s^2 - 2s\sqrt{3} - 12 = 0 \implies s = \sqrt{3} \pm \sqrt{15}.$$

Discarding the negative solution, $s = \sqrt{15} + \sqrt{3}$, so the answer is $\boxed{\textbf{(A) } 18}$.

Problem 18:

Let a and b be positive real numbers. Benny is asked to calculate the values $\log\left(\frac{a}{b}\right)$ and $\log(a \times b)$, but when solving, he mistakenly simplifies the expressions to $\frac{\log(a)}{\log(b)}$ and $\log(a) \times \log(b)$, respectively. If each of the (incorrect) answers that Benny gets is 3 greater than its corresponding correct answer, what is the maximum possible value of $\log(b)$?

(A) $2+\sqrt{7}$ **(B)** $3+\sqrt{3}$ **(C)** $2+2\sqrt{2}$ **(D)** 5 **(E)** $3+\sqrt{5}$

Proposed by ihatemath123[a]

[a] I proposed this problem, in the sense that I have manifested this situation into reality far too many times. - ihatemath123

By the given information and log properties,

$$\log\left(\frac{a}{b}\right)+3=\frac{\log(a)}{\log(b)} \implies \log(a)-\log(b)+3=\frac{\log(a)}{\log(b)}$$
$$\log(ab)+3=\log(a)\cdot\log(b) \implies \log(a)+\log(b)+3=\log(a)\cdot\log(b).$$

Let $x=\log(a)$ and $y=\log(b)$ so that the system becomes

$$x-y+3=\frac{x}{y}$$
$$x+y+3=xy$$

Multiplying the first equation by y and rearranging terms implies

$$xy=y^2+x-3y.$$

Substituting for xy in the second equation implies

$$\begin{aligned} x+y+3&=y^2+x-3y\\ y+3&=y^2-3y\\ y^2-4y-3=0 \implies y&=2\pm\sqrt{7}. \end{aligned}$$

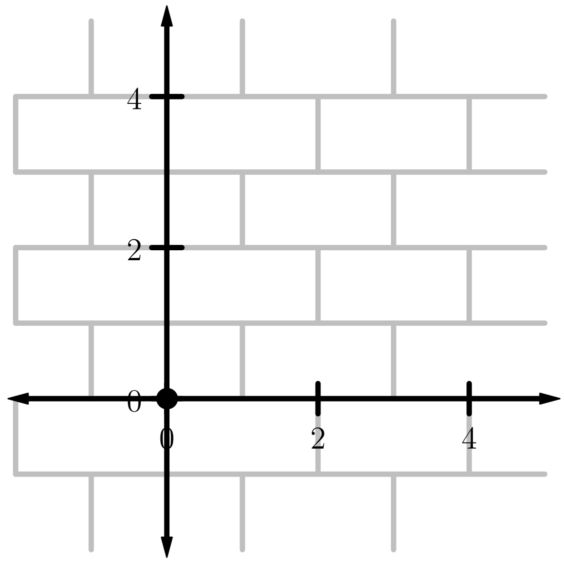
Since $y=\log(b)$, its maximum value is $\boxed{\textbf{(A) } 2+\sqrt{7}}$.

Problem 19:

An infinite brick pattern is drawn on the coordinate plane, as shown below. If a line segment is drawn from the origin to $(49, 52)$, how many bricks will this line segment pass through the interior of?

(A) 65 **(B)** 66 **(C)** 67 **(D)** 68 **(E)** 69

Proposed by ihatemath123



The line segment enters a new brick precisely when it reaches a vertical or horizontal border (or both at the same time).

The points where a vertical border and horizontal border meet are all lattice points. Since 49 and 52 are relatively prime, the line segment will not pass through a lattice point between $(0, 0)$ and $(49, 52)$, and thus, will never pass through two borders at once. Let h and v are the number of horizontal and vertical borders, respectively, that the line segment passes through. Then, the number of bricks the line segment passes through is $1 + h + v$, where the 1 accounts for the starting brick.

The horizontal borders the line segment passes through are $y = 1, 2, 3, \cdots, 51$, implying $h = 51$.

Discounting lattice points, the vertical borders occur when x and $\lfloor y \rfloor$ have opposite parities, where x is an integer and y is not necessarily an integer. (For a real number r, the function $\lfloor r \rfloor$ denotes the greatest integer less than or equal to r.)

The line segment has equation $y = \frac{52}{49}x$. Thus, v is equal to number of integers $1 \le x \le 48$ such that x and $\lfloor \frac{52}{49}x \rfloor$ have different parities. Since x is an integer, $\lfloor \frac{52}{49}x \rfloor = \lfloor (1 + \frac{3}{49})x \rfloor = x + \lfloor \frac{3}{49}x \rfloor$, so v is equivalent to the number of integers $1 \le x \le 48$ such that $\lfloor \frac{3}{49}x \rfloor$ is odd.

- If $1 \le x \le 16$, then $\lfloor \frac{3}{49}x \rfloor = 0$, which is not odd.
- If $17 \le x \le 32$, then $\lfloor \frac{3}{49}x \rfloor = 1$, which is odd.
- If $33 \le x \le 48$, then $\lfloor \frac{3}{49}x \rfloor = 2$, which is not odd.

Thus, $v = 16$, implying $1 + h + v = \boxed{\textbf{(D)}\ 68}$.

Problem 20:

For each positive integer divisor d of a positive integer N, the number of divisors of d is written down. If the numbers 4, 6 and 8 are written down 34, k and 100 times, respectively, how many possible values of k are there?

(A) 3 **(B)** 6 **(C)** 10 **(D)** 12 **(E)** 15

Proposed by ihatemath123

Let $f(n)$ be the number of primes in the prime factorization of N with an exponent of at least n. Obviously, f is non-increasing.

Any positive integer with 4 divisors is in the form p^3 or pq for distinct primes p and q. There are $f(3)$ divisors of N in the first form and $\binom{f(1)}{2}$ divisors of N in the second form. Thus, by the given information,

$$f(3) + \binom{f(1)}{2} = 34.$$

Note that $f(1) \geq f(3)$ by the non-increasing behavior of f.

- If $f(1) \leq 7$, then $f(3) \geq 34 - \binom{7}{2} = 13$, which is a contradiction.
- If $f(1) \geq 9$, then $\binom{f(1)}{2} \geq 36 > 34$, which is a contradiction.

Hence, $f(1) = 8$, implying $f(3) = 6$.

Any positive integer with 8 divisors is in the form is in the form p^7, p^3q, or pqr, where p, q, and r are distinct primes. There are $f(7)$ divisors of N in the first form. To construct a divisor of N in the second form, there are $f(3)$ choices for p and then $f(1) - 1$ choices for q. There are $\binom{f(1)}{3}$ divisors of N in the third form. Thus, by the given information

$$f(7) + f(3)(f(1) - 1) + \binom{f(1)}{3} = 100 \implies f(7) + 6 \cdot 7 + \binom{8}{3} = 100 \implies f(7) = 2.$$

Any positive integer with 6 divisors is in the form is in the form p^5 or p^2q for distinct primes p and q. There are $f(5)$ divisors of N in the first form. To construct a divisor of N in the second form, there are $f(2)$ choices for p and then $f(1) - 1$ choices for q. Hence,

$$k = f(5) + f(2)(f(1) - 1) = f(5) + 7f(2).$$

Since f is non-increasing,

$$f(1) \geq f(2) \geq f(3) \implies 6 \leq f(2) \leq 8$$
$$f(3) \geq f(5) \geq f(7) \implies 2 \leq f(5) \leq 6.$$

The value of k is unique for each choice of $(f(2), f(5))$, since each choice of $f(5)$ uniquely determines $k \pmod 7$. Thus, the number of possible k is $3 \cdot 5 = \boxed{\textbf{(E) } 15}$.

Problem 21:

A unit cube is removed from a corner of a cube with a side length of 2 units. Two points are chosen randomly and uniformly within the interior of the new solid. What is the probability that the line segment connecting these two points lies completely within the interior of this solid?

(A) $\frac{39}{49}$ **(B)** $\frac{40}{49}$ **(C)** $\frac{41}{49}$ **(D)** $\frac{43}{49}$ **(E)** $\frac{44}{49}$

Proposed by ihatemath123

Solution 1: [3]

[3] im too lazy to write this one up but yea basically casework bash on where each points lands and then exploit symmetry, basically just 2023 aime ii p6 - ihatemath123

Solution 2, due to SenorSloth: We first solve a similar problem: if the entire cube was still there, and we randomly pick two points within the cube, what's the probability that the segment we choose passes through the unit cube that would have been removed? (Let's call the cube that would have been removed the "special cube.")

Let the answer to this problem be x. Recall the formula

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

The answer to the original problem is

$$1 - P(\text{the segment passes through the special cube, given that neither point lies inside the special cube}).$$

There is a $1 - \left(\frac{7}{8}\right)^2 = \frac{15}{64}$ chance that at least one of the randomly chosen points lies *inside* the special cube, in which case the segment will obviously pass through the special cube. Thus, the probability that the segment passes through the special cube, but neither of the points lies inside the special cube, is $x - \frac{15}{64}$. This is $P(A \cap B)$. Also, there is a $\frac{7}{8}$ chance a point lies outside the special cube, so $P(B) = \left(\frac{7}{8}\right)^2 = \frac{49}{64}$.

Therefore,

$$P(A|B) = \frac{\left(x - \frac{15}{64}\right)}{\left(\frac{49}{64}\right)}$$

We can compute x in a clever way, by computing the expected number of cubes that the segment will pass through. The 8 smaller unit cubes inside the 2 by 2 by 2 square (one of which is the special cube) are separated by three "walls" that each split the cube in half.

Claim: The number of unit cubes that the segment passes through is just the number of walls that this segment passes through, plus one.

Imagine drawing the segment from one endpoint to the other. Every single time we pass through a wall, we enter a new cube. So if we pass through n walls, we also enter n new cubes. Adding the initial cube we start at, passing through n walls corresponds to passing through $n + 1$ cubes.

For each wall, the segment passes through it iff the second point lies on the opposite side of the wall as the first point, which happens with probability $\frac{1}{2}$. Therefore, by linearity of expectation, the segment is expected to pass through $\frac{3}{2}$ walls, and $\frac{5}{2}$ cubes. There are 8 cubes in total, so the probability that the segment passes through any given cube is

$$\frac{\left(\frac{5}{2}\right)}{8} = \frac{5}{16}.$$

The special cube is a given cube! Therefore, $x = \frac{5}{16}$, so our final answer is

$$1 - \frac{\left(x - \frac{15}{64}\right)}{\left(\frac{49}{64}\right)} = \boxed{\textbf{(E) } \frac{44}{49}}.$$

[4]

[4] This is a bit longer and less simple than the first solution, but it's made infinitely better by the fact that there is no casework; the reasoning here seamlessly generalizes to higher dimensions. It's also a really cool combination of a bunch of random probability techniques. - ihatemath123

Problem 22:

Let ℓ and m be two intersecting lines, and let θ be the acute angle formed by their intersection. Points A and B exist, coplanar with the two lines, such that the distances from A to ℓ, A to m, B to ℓ, B to m and A to B are 10, 3, 4, 8 and 7, respectively. There are two possible values of θ; let these values be α and β. What is $\sin(\alpha+\beta)$?

(A) $\frac{19\sqrt{6}}{49}$ **(B)** $\frac{33\sqrt{2}}{49}$ **(C)** $\frac{18\sqrt{7}}{49}$ **(D)** $\frac{34\sqrt{2}}{49}$ **(E)** $\frac{20\sqrt{6}}{49}$

Proposed by ihatemath123

For a point P and line L, define $d(P, L)$ as the distance from P to L.

If A and B lied on opposite sides of ℓ, then

$$AB \geq d(A, \ell) + d(B, \ell) = 14,$$

but this contradicts $AB = 7$. Similarly, if A and B lied on opposite sides of m, then

$$AB \geq d(A, m) + d(B, m) = 11,$$

but this contradicts $AB = 7$.

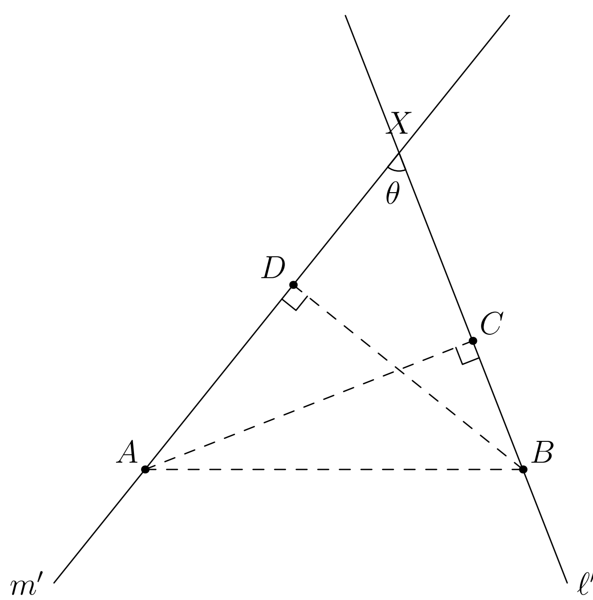
Thus, A and B lie on the same side of ℓ and on the same side of m.

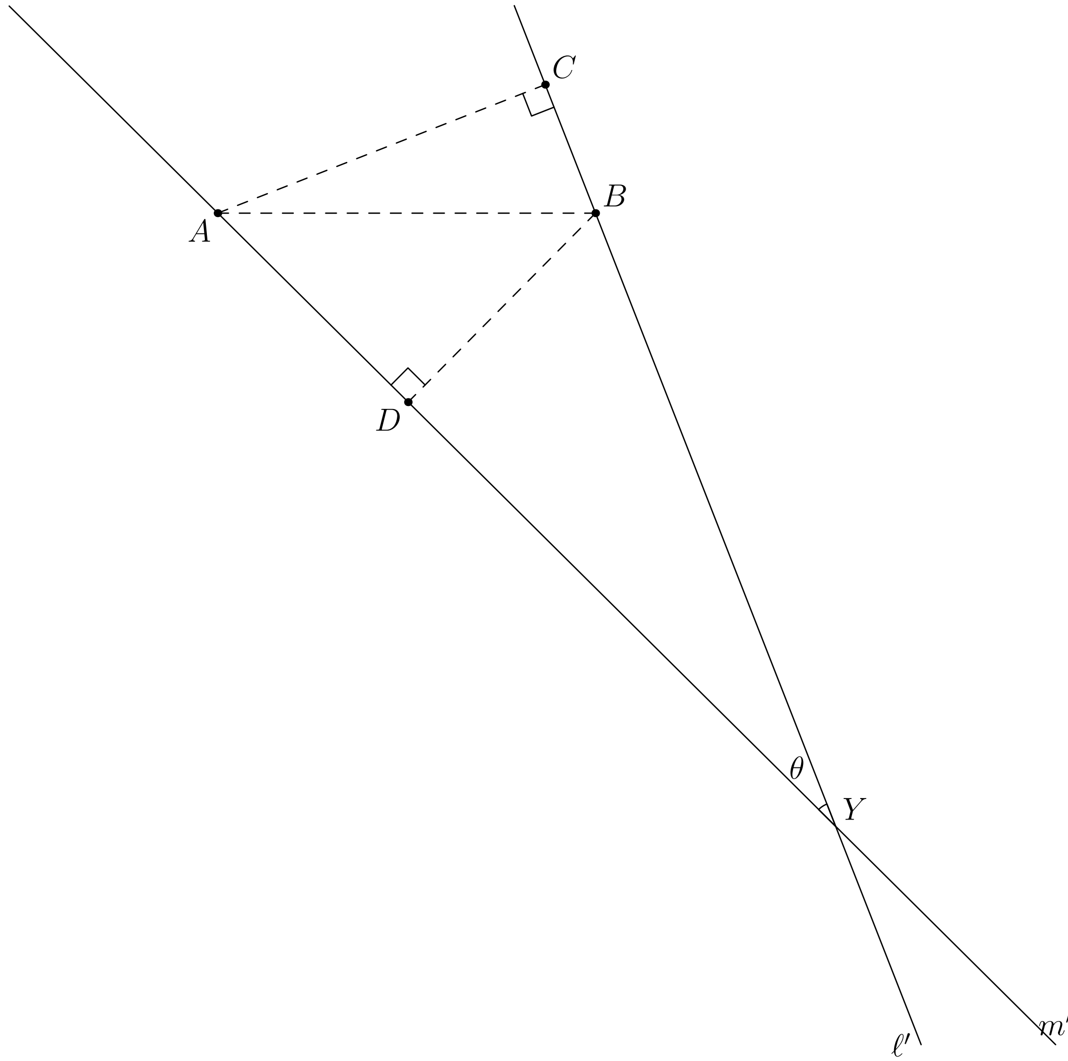
Define line ℓ' as the line parallel to ℓ, passing through B. Then, $d(A, \ell') = 10 - 4 = 6$. Similarly, define line m' as the line parallel to m passing through A. Then, $d(B, m') = 8 - 3 = 5$.

Since $\ell \parallel \ell'$ and $m \parallel m'$, the angle between the two lines is preserved. Thus, it remains to find all possible acute angles θ formed by the intersection of ℓ' and m'.

Let C be the foot of the altitude from A to ℓ' and D as the foot of the altitude from B to m'. Then, $AC = 6$ and $BD = 5$. Also, θ is equal to the acute angle formed by lines AD and BC.

There are two configurations. Either C and D are on the same side of AB, or they are on opposite sides.





For rigor, we justify some facts about how these two diagrams were drawn:

- In the first configuration, X does not lie inside $ABCD$. From the Pythagorean theorem, we have that $\sqrt{24} = AD < DB = \sqrt{25}$, so D lies to the left of the perpendicular bisector of $\overline{AB}$. On the other hand, by similar reasoning, C lies to the right of the perpendicular bisector of $\overline{AB}$. Therefore, no point lies on segments $\overline{AD}$ and $\overline{BC}$, so the intersection of lines AD and BC lies outside of $ABCD$.

- In the first configuration, $\angle AXB < 90°$. Since $AD > DB$, we have $\angle A > 45°$. By similar reasoning, $\angle B > 45°$. So, $\angle AXB < 180° - 45° - 45° = 90°$.

- In the second configuration, Y lies on the opposite side of line AB as C. Note that $\angle CBA < \angle BAD$ [5], so $180° > 180° - \angle CBA + \angle BAD = \angle ABY + \angle BAY$. This means that rays AD and CB intersect.

- In the second configuration, $\angle AYB < 90°$. Clearly, $\angle ABY$ is obtuse.

Note that at this point, α and β could be painstankingly computed via "trigbash". However, we don't need to find α and β; we just need to find $\sin(\alpha + \beta)$, and there is a weird but funny way to do so.[6]

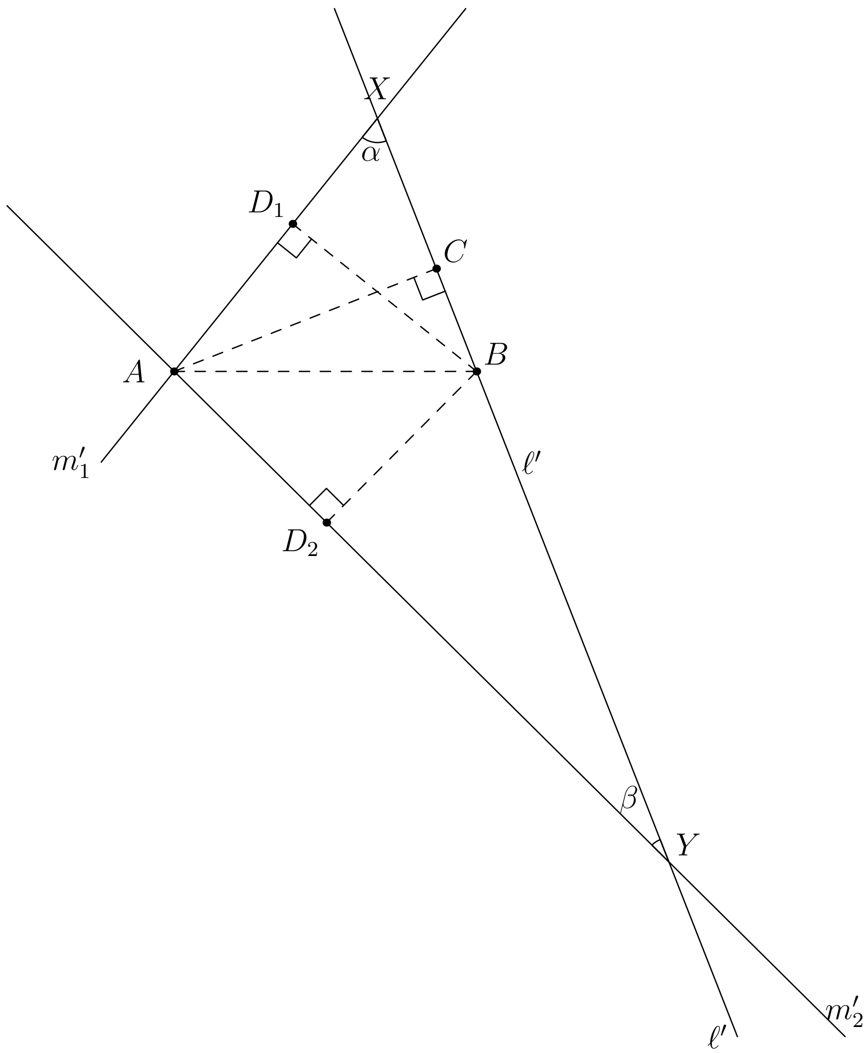
Imagine if we printed the two configurations on two transparent plastic sheets. Then, we could place one sheet on top of the other so that in both diagrams, points A, B and C coincided. Then, clearly, ℓ' would coincide too in both diagrams. Therefore, X, B and Y would be collinear, so α and β would be two of the

[5] left as an excercise to reader - ihatemath123

[6] Oops the problem kinda sucks from here; when writing this problem, at first, I didn't realize that there were two separate configurations, so initially the extraction just asked for $\sin\theta$ with some nice numbers given. I still think the (very unmotivated) solution that follows is pretty cool. - ihatemath123

angles in $\triangle AXY$; the third angle would be $\angle XAY$. So,

$$\begin{aligned} \alpha+\beta &= 180^\circ - \angle XAY \\ \implies \sin(\alpha+\beta) &= \sin(\angle XAY) \\ &= \sin(2\angle DAB) \\ &= 2\sin(\angle DAB)\cos(\angle DAB) \\ &= 2\cdot\frac{5}{7}\cdot\frac{\sqrt{24}}{7} \\ &= \boxed{\textbf{(E) } \frac{20\sqrt{6}}{49}}. \end{aligned}$$



Problem 23:

Let a, b, c, d and e be reals such that

$$\begin{cases} \frac{a}{b}+\frac{b}{c} & =1 \\ \frac{b}{c}+\frac{c}{d} & =2 \\ \frac{c}{d}+\frac{d}{e} & =3 \\ \frac{d}{e}+\frac{e}{a} & =4. \end{cases}$$

What is the sum of the *squares* of the 5 possible values of

$$\frac{e}{a}+\frac{a}{b}?$$

(A) 30 **(B)** 40 **(C)** 50 **(D)** 60 **(E)** 70

Proposed by ihatemath123

Define $r_1=\frac{a}{b}$, $r_2=\frac{b}{c}$, $r_3=\frac{c}{d}$, $r_4=\frac{d}{e}$, $r_5=\frac{e}{a}$ and let $k=\frac{e}{a}+\frac{a}{b}$. Then,

$$\begin{aligned} r_1+r_2&=1\\ r_2+r_3&=2\\ r_3+r_4&=3\\ r_4+r_5&=4\\ r_5+r_1&=k. \end{aligned}$$

We've dealt with the four-equation-five-variable problem, because we actually have a fifth equation too:

$$r_1r_2r_3r_4r_5=\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{d}\cdot\frac{d}{e}\cdot\frac{e}{a}=1.$$

Summing and halving the five sums yields that $r_1+r_2+r_3+r_4+r_5=\frac{k}{2}+5$, and subtracting pairs of equations gives us $(r_1,r_2,r_3,r_4,r_5)=(\frac{k}{2}-1,-\frac{k}{2}+2,\frac{k}{2},-\frac{k}{2}+3,\frac{k}{2}+1)$. But $r_1r_2r_3r_4r_5=1$, so

$$\left(\frac{k}{2}-1\right)\left(-\frac{k}{2}+2\right)\left(\frac{k}{2}\right)\left(-\frac{k}{2}+3\right)\left(\frac{k}{2}+1\right)=1.$$
$$k(k-2)(k-4)(k-6)(k+2)-32=0.$$

We are asked for the sum of the squares of k. This can be done by expanding the above polynomial and using Vieta's formulas; however, there is a clever trick we can use to speed up the computation: note that the constant term in this polynomial will not affect the sum of the square of its roots. Therefore, the sum of the squares of the roots of this polynomial are equal to the sum of the squares of the roots of

$$k(k-2)(k-4)(k-6)(k+2)=0,$$

which is $(-2)^2+0^2+2^2+4^2+6^2=\boxed{\textbf{(D) } 60}$.

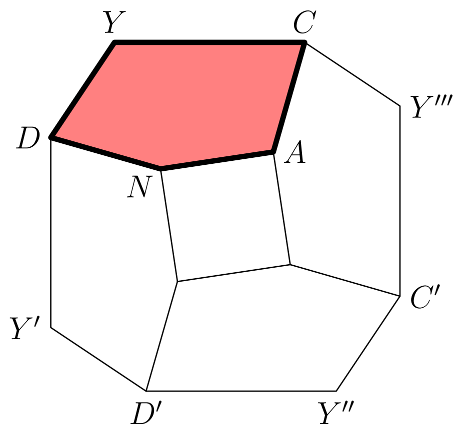
Problem 24:

Convex pentagon $CANDY$ has side lengths $CA = AN = ND = DY = 6$ and $YC = 10$. If $\angle A + \angle N = \angle D + \angle Y + \angle C$, what is the largest possible integer area of pentagon $CANDY$?

(A) 61 **(B)** 63 **(C)** 65 **(D)** 67 **(E)** 69

Proposed by ihatemath123

The condition we have is just $\angle A + \angle N = 270^\circ$. Because of this, and the fact that $CA = ND = 6$, we can rotate our pentagon four times by 90° and glue rotations together like this:



Note that $[CANDY] = \frac{\text{(Area of octagon)}-36}{4}$, where 36 is the area of the center square. So, we just wish to maximize the area of the octagon. Letting $\angle CYD = \theta$, we see that $[CYD] = [CY'''C'] = [C'Y''D'] = [D'Y'D] = 30\sin(\theta)$, and that $[DCC'D'] = CD^2 = 136 - 120\cos(\theta)$. So,

$$[YCY'''C'Y''D'Y'D] = 136 - 120\cos(\theta) + 120\sin(\theta).$$

The maximum possible value of $\sin(\theta) - \cos(\theta)$ is $\sqrt{2}$ obtained when $\theta = 135^\circ$, so the maximum possible area of the octagon is $136 + 120\sqrt{2}$.[7] Then, the maximum possible value of $CANDY$ is

$$\frac{136 + 120\sqrt{2} - 36}{4} = 25 + 30\sqrt{2} \approx 67.4,$$

so the maximum possible integer value is $\boxed{\textbf{(D)}\ 67}$.

[7] Alternatively, one can utilize the fact that, given the side lengths of a polygon, its area is maximized when the polygon is configured to be cyclic.

Problem 25:

Carlos the Caterpillar is at the origin. When he is at the point (x, y), he can wiggle to either $(x+1, y)$ or $(x, y+1)$. Call a point whose coordinates are both odd integers an *odd point*. There are N paths that Carlos can take to get to the point $(16, 16)$ that pass through 14 odd points. How many positive divisors does N have?

(A) 64 **(B)** 72 **(C)** 80 **(D)** 88 **(E)** 96

Proposed by ihatemath123

Let $\mathcal{O}$ be the set of odd points such that $0 \leq x, y \leq 16$. Then, $\mathcal{O}$ is a 8×8 lattice. From any odd point in Carlos's path, the next odd point he goes to clearly cannot be anywhere to the left or anywhere below the previous odd point. It becomes clear Carlos cannot pass through more than 15 odd points.

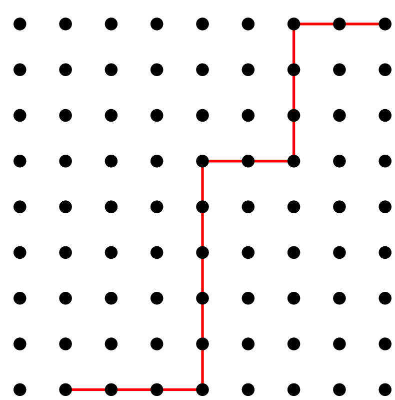
Let S be the subset of points in $\mathcal{O}$ that Carlos passes through. In addition, let P be a path of the points in $\mathcal{O}$ starting at $(1, 1)$ and ending $(15, 15)$ such that the next point in the path is directly right or directly up of the previous point. All paths P pass through exactly 15 points in $\mathcal{O}$. Thus, S can only be formed by taking some path P and deleting one of the points in the path.

The deleted point cannot be the middle point of some three consecutive points that are collinear as Carlos would have no way to go around that point. Thus, the deleted point is either $(1, 1)$, $(15, 15)$, or a point where when the path bends.

Case 1: The deleted point is either $(1, 1)$ or $(15, 15)$.

Without loss of generality, assume that the deleted point is $(1, 1)$. Multiply by 2 at the end of the case.

The first odd point Carlos visits can either be $(3, 1)$ or $(1, 3)$ (both create the same number of valid paths by symmetry across $y = x$). Without loss of generality, assume the first odd point Carlos visits is $(3, 1)$. Multiply by 2 at the end of the case. An example of S in this case is shown below with the connected dots (where the dots represent $\mathcal{O}$):



Considering the steps of the odd points, all such examples of S consist of 6 moves to the right and 7 moves up, which can be ordered in $\binom{13}{6}$ ways.

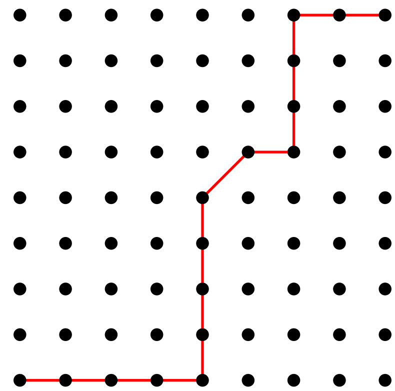
Now, the odd points in S must be used to construct a valid path.

- First, Carlos must get from $(0, 0)$ to $(3, 1)$ without passing through $(1, 1)$. This can be done if he goes to $(2, 0)$ then moves up then right, or right then up. There are 2 ways for Carlos to do this step.
- Then, Carlos must get to $(15, 15)$ while passing through the correct number of odd points. As computed before, this can be done in $\binom{13}{6}$ ways.
- Lastly, Carlos must get to $(16, 16)$. Carlos can either move up then right, or right then up. There are 2 ways for Carlos to do this step.

Thus, the total for this case is $2 \cdot 2 \cdot (2 \cdot \binom{13}{6} \cdot 2) = 16\binom{13}{6}$.

Case 2: The deleted point is where a path bends

An example of S in this case is shown below with the connected dots (note the diagonal move):



Considering the steps of the odd points, all such examples of S consist of 6 steps to the right, 6 steps up, and 1 diagonal step. These steps can be ordered in $\frac{13!}{6!6!1!} = 7\binom{13}{6}$ ways.

Now, the odd points in S must be used to construct a valid path.

- First, Carlos must get from $(0,0)$ to $(1,1)$. This can be done in 2 ways.
- Next, Carlos must get to $(15,15)$ while passing through the correct number of odd points. The odd points can be chosen in $7\binom{13}{6}$ ways.
- For the unique diagonal move, Carlos can perform it in more than one way. Suppose the two odd points in the diagonal move are (x,y) and $(x+2,y+2)$. To prevent Carlos from passing through other odd points, he must go to $(x+1,y+1)$, which can be done in 2 ways. From there, he can go to $(x+2,y+2)$ in 2 ways. So there are 4 ways he can perform the diagonal move.
- From $(15,15)$, he must go to $(16,16)$, which can be done in 2 ways.

Thus, the number of paths in this case is $2 \cdot 7\binom{13}{6} \cdot 4 \cdot 2 = 112\binom{13}{6}$.

Combining the cases,

$$N = 16\binom{13}{6} + 112\binom{13}{6} = 128\binom{13}{6} = 2^7 \cdot (2^2 \cdot 3 \cdot 11 \cdot 13) = 2^9 \cdot 3 \cdot 11 \cdot 13.$$

The number of divisors of N is $(9+1)(1+1)(1+1)(1+1) = \boxed{\textbf{(C) } 80}$.