ZeMC 10A Results and Solutions

Ze Committee

Thursday, November 24th

This document contains results & solutions for the ZeMC 10A, a mock AMC 10 held for the 2022-2023 competition season. The test can be found here. If you have any questions, you should contact ihatemath123 on AoPS, or imagine dragon#3311 on Discord. You can also check the ZeMC public discussion forum.

(Disclaimer from ihatemath123: I couldn't get the diagrams working in the problem statements, so the statements are diagramless.)

The ZeMC competition series is made possible by the contributions of the following problem-writers and test-solvers:

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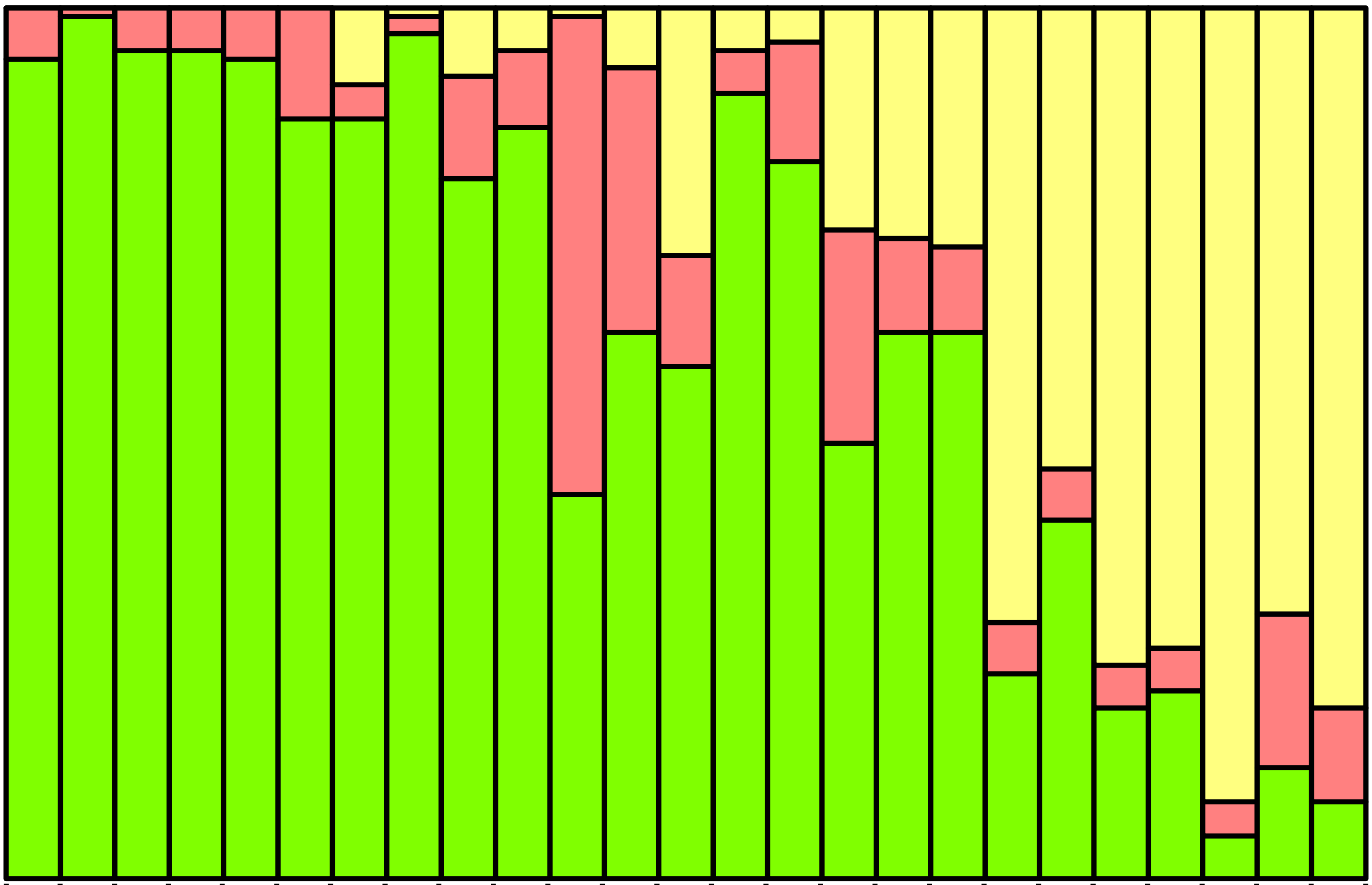
This solutions manual was written by

P_Groudon.

Answer Key and Solve Rates

The chart of solve rates below reflects the number of correct, incorrect and blank answers given for each problem on the contest. In the chart, green denotes a correct answer, red denotes an incorrect answer and yellow denotes no answer.



1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
C	B	D	B	B	D	B	B	D	D	B	E	E	A	D	C	B	E	B	E	A	A	D	E	E

Problem 1:

What is

$$2^{\left(0^{\left(2^2\right)}\right)}+2^{\left(2^{\left(0^2\right)}\right)}+2^{\left(2^{\left(2^0\right)}\right)}?$$

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Proposed by ihatemath123

Claim: For a function f with domain integers at least -1, and $f(-1)=0$ and $f(n)=2^{f(n-1)}$ for $n\geq 0$, f is injective.

Note that $2^x > x$ for all real x, so f is increasing and therefore injective.

Claim: If one of the twos is replaced by a zero in a power tower of n twos, the result is different for each position the replacement was done in.

Suppose that we replaced the ath two from the bottom with a zero in $2^{2^{\cdots}}$. Then, it becomes a tower of $a-2$ twos topped off with a $2^0=1$, so the value of the expression is $f(a-2)$. Since we know that f is injective, a different value is obtained for each a.

Note that each of the three terms is a power of 2, and as shown by our second claim, are distinct. Note that a positive integer n can be represented as the sum of 3 distinct powers of 2 if and only if its sum of digits in binary, which we call $s_2(n)$, is equal to 3. We use the formula $s_2(n)=n-v_2(n!)$ using p-adic notation.

We then compute

$$\begin{aligned} v_2(120)&=3\\ v_2(720)&=4\\ v_2(5040)&=4\\ v_2(40320)&=7\\ v_2(362880)&=7. \end{aligned}$$

This gives $s_2(5)=2$, $s_2(6)=2$, $s_2(7)=3$, $s_2(8)=1$, and $s_2(9)=2$. Since the sum of the digits in binary of the answer must be 3, our answer is $\boxed{\textbf{(C) } 7}$.[1]

Problem 2:

The "?" in the equation below represents a single nonzero digit. What is that digit?

(A) 1 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 9

Proposed by bissue

9? - ?5 = ?9

By analyzing the units digit of the equation, the "?" must be a $\boxed{\textbf{(B) } 4}$. Indeed, the equation

$$94-45=49,$$

is true.

[1] lol

Problem 3:

How many positive integers less than 1000 can be written as the product of three consecutive integers?

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

Proposed by ihatemath123

The only working numbers are $1 \times 2 \times 3$, $2 \times 3 \times 4$, $\dots$, $9 \times 10 \times 11$, for an answer of $\boxed{\textbf{(D)}\ 9}$. (The numbers $10 \cdot 11 \cdot 12$ and beyond are greater than 1000, and the numbers $0 \cdot 1 \cdot 2$ and unbeyond are not positive.)

Problem 4:

Steven walks at a rate of 1.5 meters per second and runs at a rate of 4 meters per second. He travels the 850 meters to school by foot in 5 minutes. How much of that time is spent walking, in seconds?

(A) 120 **(B)** 140 **(C)** 160 **(D)** 180 **(E)** 200

Proposed by bissue

Suppose Steven spends x seconds walking and y seconds running. Then, $x + y = 300$, since the total time is 5 minutes.

Using the formula $d = rt$, the distance he travels by walking in meters is $1.5x$, while the distance he travels by running in meters is $4y$. Thus, $1.5x + 4y = 850$.

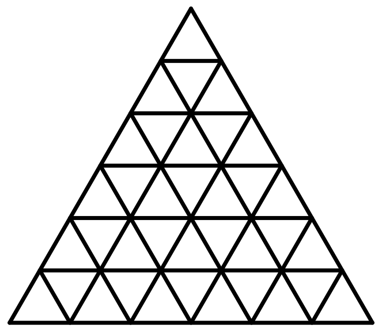
Solving the system of equations implies $x = 140$ and $y = 160$. Thus, the amount of time spent walking in seconds is $\boxed{\textbf{(B)}\ 140}$.

Problem 5:

The diagram below consists of 36 equilateral triangles placed side by side. How many regular hexagons are there in the diagram?

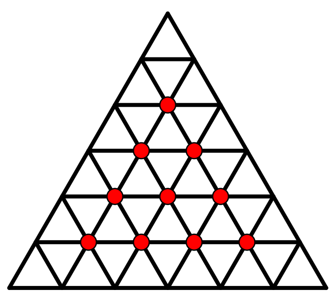
(A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

Proposed by ihatemath123

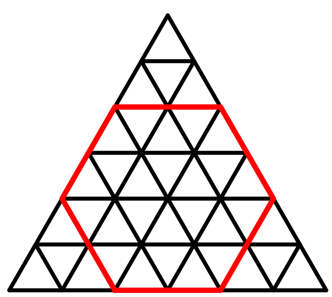


Consider cases on the side length of the regular hexagon, where 1 unit is equal to the length of one of the 36 equilateral triangles.

If the hexagon has side length 1, there are 10 such hexagons, where the red points indicate possible centers for the hexagon:



If the hexagon has side length 2, there is only 1 such hexagon:



It is also clear that there cannot be a bigger hexagon in the diagram. Thus, the total number is $10+1 = \boxed{\textbf{(B) } 11}$.

Problem 6:

What is the minimum possible mean of a set of 6 distinct positive primes whose median is 22?

(A) $20\frac{1}{6}$ **(B)** $20\frac{1}{2}$ **(C)** $20\frac{5}{6}$ **(D)** $21\frac{1}{6}$ **(E)** $21\frac{1}{2}$

Proposed by ihatemath123

Let $p_1, p_2, \cdots, p_6$ be the primes in increasing order. The mean is minimized when the sum $p_1 + p_2 + \cdots + p_6$ is minimized. Since the median is 22, it follows that $p_3 + p_4 = 44$. To minimize the sum, p_4 should be minimized to allow p_5 and p_6 to be smaller.

Since $p_3 < p_4$, it follows that $p_4 > 22$; from testing primes, we see that the minimum possible value of p_4 is achieved by setting $p_3 = 13$ and $p_4 = 31$. Then, to minimize the sum, set $p_1 = 2$ and $p_2 = 3$ (the two smallest primes), while $p_5 = 37$ and $p_6 = 41$ (the two smallest primes greater than 31). The mean is

$$\frac{2 + 3 + 13 + 31 + 37 + 41}{6} = \boxed{\textbf{(D) } 21\frac{1}{6}}.$$

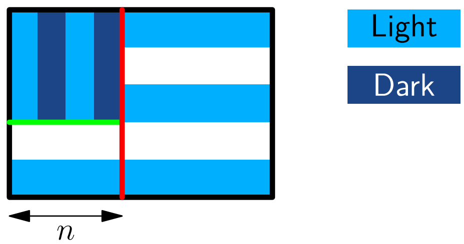
Problem 7:

The flag of New Zeland consists of five evenly spaced horizontal stripes, alternating between light blue and white, and a rectangle in the top left corner, filled with 4 evenly spaced vertical stripes, alternating between light blue and dark blue. If 53% of the entire flag is colored light blue, what percent of the flag is colored dark blue?

(A) 20% **(B)** 21% **(C)** 22% **(D)** 23% **(E)** 24%

Proposed by ihatemath123

Solution 1: Without loss of generality, assume that the width of the flag is 1 and the height is h. Let n be a fraction of the rectangle's width as shown below.



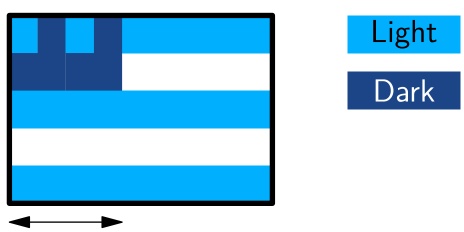
There are two rectangles left of the red line: one above and one below the green line. It is clear in both rectangles, exactly half of the area is covered by light blue. Thus, the light blue area to the left of the red line is $\frac{1}{2}nh$.

To the right of the red line, it is clear exactly $\frac{3}{5}$ths of the area is covered by light blue. Thus, the light blue area to the right of the red line is $\frac{3}{5}(1-n)h$. Since the total light blue area is given to be $0.53h$,

$$\frac{1}{2}n + \frac{3}{5}(1-n) = 0.53 \implies n = 0.7.$$

The dark blue area in the flag is given by half of the area of the rectangle left of the red line and above the green line. Since this rectangle has 0.7 of the width of the flag and $\frac{3}{5}$ths of its height, the request percentage is $\frac{1}{2} \cdot 0.7 \cdot \frac{3}{5} = \boxed{\textbf{(B) } 21\%}$.

Solution 2, due to Rohan Garg: We can rearrange parts of the top-left corner while preserving the amount of dark blue, light blue and white:



Almost three fifths of the flag is colored light blue; those two dark blue rectangles at the top invade light blue's territory, cutting light blue down from what would be 60%, to a measly 53%. This means that those two dark blue rectangles at the top make up 7% of the whole flag. There are six of these dark blue rectangles in total, so dark blue covers $7\% \times 3 = \boxed{\textbf{(B) } 21\%}$ of the flag.

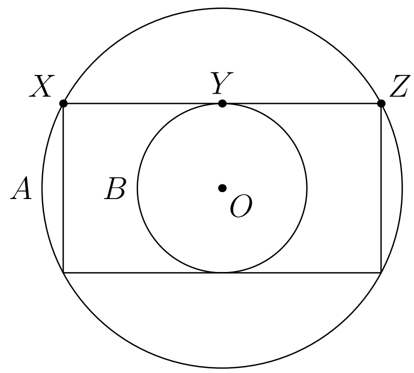
Problem 8:

A rectangle R is inscribed in a circle A, and a circle B is the largest possible circle that fits within rectangle R. If the area of circles A and B are 15π and 3π respectively, what is the area of rectangle R?

(A) 18 **(B)** 24 **(C)** 30 **(D)** 36 **(E)** 42

Proposed by ihatemath123

Since circle B can be freely translated horizontally within the confines of R, let B share its center with A. Point labels are shown:



By the given information about the areas of the circles, it follows that $OX = \sqrt{15}$ and $OY = \sqrt{3}$. Since B is tangent to XZ at Y, $\angle OYX = 90^\circ$. Thus, by Pythagorean Theorem, $XY = 2\sqrt{3}$. By symmetry, Y is the midpoint of $\overline{XZ}$, so $XZ = 4\sqrt{3}$. The height of the rectangle is twice the radius of B or $2\sqrt{3}$. Thus, the area of R is $4\sqrt{3} \cdot 2\sqrt{3} = \boxed{\textbf{(B) } 24}$.

Problem 9:

Emily and Ruby are two students among a class of 21. Their teacher will randomly divide the class into 4 groups of 4 and one group of 5 for an upcoming project. What is the probability that Emily and Ruby are put in the same group?

(A) $\dfrac{16}{105}$ **(B)** $\dfrac{68}{441}$ **(C)** $\dfrac{4}{25}$ **(D)** $\dfrac{17}{105}$ **(E)** $\dfrac{21}{125}$

Proposed by ihatemath123

Suppose Emily is randomly assigned to a group first and then Ruby is randomly assigned to a group. There are 2 cases on whether Emily and Ruby are in a group of 4 or group of 5.

1. Emily is in any one of the groups of 4 with probability $\frac{16}{21}$. Then, the probability that Ruby is in the same group of 4 Emily is in is $\frac{3}{20}$.

2. Emily is in the group of 5 with probability $\frac{5}{21}$. Then, the probability that Ruby is in group of 5 with Emily is in is $\frac{4}{20}$.

Thus, the desired probability is $\frac{16}{21} \cdot \frac{3}{20} + \frac{5}{21} \cdot \frac{4}{20} = \boxed{\textbf{(D) } \dfrac{17}{105}}$.

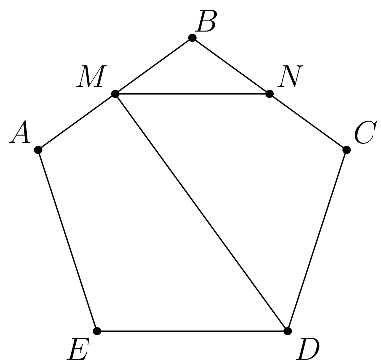
Problem 10:

In regular pentagon $ABCDE$, let M and N be the midpoints of $\overline{AB}$ and $\overline{BC}$, respectively. What is $\angle DMN$?

(A) 36° **(B)** 42° **(C)** 48° **(D)** 54° **(E)** 60°

Proposed by ihatemath123

See the diagram below:



Since M is the midpoint of $\overline{AB}$, the pentagon is symmetric about $\overline{MD}$, so $\angle DMB = 90^\circ$.

Because the pentagon is regular, $AB = BC$, implying that $BM = BN$. Thus, $\triangle BMN$ is isosceles. Since the interior angle of a regular pentagon is 108°, it follows that $\angle BMN = \angle BNM = 36^\circ$.

Thus, $\angle DMN = \angle DMB - \angle BMN = \boxed{\textbf{(D)}\ 54^\circ}$.

Problem 11:

How many ordered pairs of positive integers (M, N) satisfy

$$(M-N)^{(M+N)} = 256?$$

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Proposed by ihatemath123

We may write 256 in the following ways:

1. 256^1. This gives us that $M - N = 256$ and $M + N = 1$, which implies that $N = -\frac{255}{2}$, contradicting the fact that M and N are positive. This gives us no solutions.
2. 16^2. This gives us that $M - N = 16$ and $M + N = 2$, which implies that $N = -7$, contradicting the fact that M and N are positive. This gives us no solutions.
3. 4^4. This gives us $M - N = 4$ and $M + N = 4$, which implies that $N = 0$, contradicting the fact that M and N are positive. This gives us no solutions.
4. 2^8. This gives us $M - N = 2$ and $M + N = 8$, which is satisfied by $M = 5$ and $N = 3$. Thus, this is one solution.
5. $(-2)^8$. This gives us that $M - N = -2$ and $M + N = 8$, which is satisfied by $M = 3$ and $N = 5$. Thus, this is one solution.
6. $(-4)^4$. This gives us that $M - N = -4$ and $M + N = 4$, which implies that $M = 0$, contradicting the fact that M and N are positive. This gives us no solutions.
7. $(-16)^2$. This gives us that $M - N = -16$ and $M + N = 2$, which implies that $M = -7$, contradicting the fact that M and N are positive. This gives us no solutions.

Only the fourth and fifth values we checked yielded solutions, so the answer is $\boxed{\textbf{(B) } 2}$.

Problem 12:

How many nonempty subsets of the first 12 positive integers contain more multiples of 3 than multiples of 2?

(A) 192 **(B)** 256 **(C)** 320 **(D)** 384 **(E)** 448

Proposed by ihatemath123

It does not matter whether 1, 5, 7 or 11 are in the subset; the inclusion/exclusion of any of these numbers does not affect the number of multiples of 2 or 3 in the subset. Also, it does not matter whether 6 or 12 are in the subset, since the inclusion/exclusion of these numbers does not affect whether there are more multiples of two than three, or vice-versa. So, we only have to count the number of subsets of $\{2, 3, 4, 8, 9, 10\}$ that contain more multiples of three than two, and multiply this by 2^6, the number of subsets of $\{1, 5, 6, 7, 11, 12\}$.

Let S_2 be the set of positive integers less than or equal to 12 that are multiples of 2 and 3 but not multiples of 6 (which covers the integers S_1 does not cover). Specifically, S_2 includes $2, 4, 8,$ and 10 as well as 3 and 9.

For $a - b > 0$, the pair (a, b) can only be $(1, 0)$, $(2, 0)$, or $(2, 1)$, given the number of multiples of 3 available. For the three pairs, the elements from S_2 can be selected in 2, 1, and 4 ways in that respective order. Hence, there are $2 + 1 + 4 = 7$ ways to choose elements from S_2 to be included in the subset. In addition, since $a > 0$ in each of these pairs, the subset will never be empty.

Including the ways to freely choose integers from S_1 to be in the subset, the total number of subsets is $7 \cdot 2^6 = \boxed{\textbf{(E) } 448}$.

Problem 13:

A test contains 5 equally weighted true/false questions, with one correct answer for each problem. Angela randomly selects an answer for each problem. When she finishes, she is told how many problems she got correct (but not which problems). Angela may then modify her answers. If she makes these modifications in a way that maximizes the expected final score on her test, to the nearest percent, what is this expected score?

(A) 61% **(B)** 63% **(C)** 65% **(D)** 67% **(E)** 69%

Proposed by ihatemath123

If Angela is told that she gets less than half of the problems correct initially, each problem is more likely to be incorrect than correct; therefore, for optimization, Angela should switch all her answers (i.e. True to False, False to True). If she changes all her answers, her score goes from n to $5-n$.

Otherwise, each problem is more likely to be correct than incorrect, so she should leave everything untouched.

If she's told that she scores, $0, 1$ or 2, then, the best she can do is score a 5, 4 or 3, respectively; if she scores a 3, 4 or 5, changing anything is more likely to decrease her score than increase it.

Using binomial probabilities, Angela will get an initial score of n with probability $\frac{1}{32} \cdot \binom{5}{n}$. So,

- Angela's final score will be a 5 with probability $\frac{1}{32} \cdot (\binom{5}{0} + \binom{5}{5})$.
- Angela's final score will be a 4 with probability $\frac{1}{32} \cdot (\binom{5}{1} + \binom{5}{4})$.
- Angela's final score will be a 3 with probability $\frac{1}{32} \cdot (\binom{5}{2} + \binom{5}{3})$.

Thus, her expected score is

$$\frac{5}{32} \cdot \left(\binom{5}{0} + \binom{5}{5}\right) + \frac{4}{32} \cdot \left(\binom{5}{1} + \binom{5}{4}\right) + \frac{3}{32} \cdot \left(\binom{5}{2} + \binom{5}{3}\right) = \frac{55}{16}.$$

In terms of a percentage, her expected score will be $\frac{55}{16 \cdot 5} = \frac{11}{16} \approx \boxed{\textbf{(E) } 69\%}$.[2]

Problem 14:

Two points A and B lie on a regular hexagon, such that $\overline{AB}$ divides the hexagon into two pentagons. Both pentagons have an area of $36\sqrt{3}$ and a perimeter of $19\sqrt{3}$. What is AB?

(A) $7\sqrt{3}$ **(B)** $5\sqrt{6}$ **(C)** $9\sqrt{2}$ **(D)** $8\sqrt{3}$ **(E)** $6\sqrt{6}$

Proposed by ihatemath123

Clearly, the total area of the hexagon will be $36\sqrt{3} \times 2 = 72\sqrt{3}$. Since the area of a regular hexagon in terms of its side length is $\frac{3s^2\sqrt{3}}{2}$, it follows that

$$\frac{3s^2\sqrt{3}}{2} = 72\sqrt{3} \implies s^2 = 48 \implies s = 4\sqrt{3}.$$

Thus, the perimeter of the hexagon is $24\sqrt{3}$. Clearly, the two pentagons' combined perimeter discounting $\overline{AB}$ covers the entirety of the perimeter of the regular hexagon exactly once. Thus, each pentagon's perimeter minus $\overline{AB}$ is half the perimeter of the regular hexagon, namely $12\sqrt{3}$. Since the perimeter of the pentagon is $19\sqrt{3}$, it follows that $AB = \boxed{\textbf{(A) } 7\sqrt{3}}$.

[2] I swear, this wasn't on purpose! The numbers in this problem, and even this percentage extraction, were all written before I computed the final answer. - ihatemath123

Problem 15:

Suppose that a, b, c, d, e and f are nonzero real numbers such that exactly one of the five products below is positive. Which one is positive?

(A) ace **(B)** bdf **(C)** cad **(D)** deb **(E)** feb

Proposed by bissue

Note that $(ace)\times(bdf) = (cad)\times(feb)$. It's impossible for three of these terms to be negative and one of them to be positive, so all four must be negative. Furthermore, by setting $a = f = -69$ and $b = c = d = e = 420$, we show that making deb positive and the other four negative is attainable, so our answer is $\boxed{\textbf{(D)}\ deb}$.

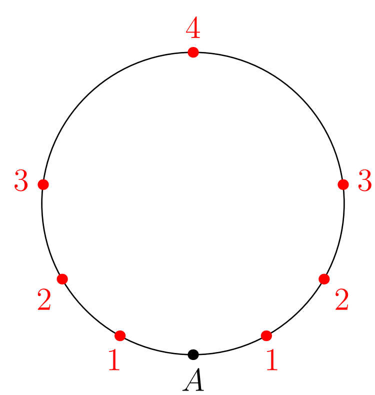
Problem 16:

Sam jogs one lap, at a constant rate, around a circular track with a radius of 10 meters, such that it takes 20 seconds for him to complete the lap. He begins jogging from a point P on the track. Every time his **straight-line** distance from P in meters is a positive integer, his coach writes down the exact number of seconds that have elapsed since he began jogging. After the lap, what is the sum of all the numbers Sam's coach wrote down?

(A) 200 **(B)** 210 **(C)** 390 **(D)** 400 **(E)** 410

Proposed by ihatemath123

Let Sam's starting point be A. A common mistake may be to assume that those points that the coach writes down are equally spaced. This is not true! For example, consider a circle with a radius of 2. Points that are an integer distance from A are marked in red:



In fact, we don't need to calculate any actual times. Just note that for $x = 1, 2, 3, \dots, 19$, there are two moments where Sam is x meters from A. At one of these points, let t be the amount of time Sam has been running for. Then, at the other point, Sam will have been running for $20 - t$ seconds. So, the sum of those two times is $20 - t + t = 20$ seconds.

There is exactly one point for which Sam is 20 meters from A. He is halfway around the track at that point, so he will have been running for 10 seconds. Our final answer is

$$19 \cdot 20 + 10 = \boxed{\textbf{(C)}\ 390}.$$

Problem 17:

Claire chooses a four digit integer N with nonzero digits, and lists all possible permutations of the digits of N (including N itself). If 12 integers in her list are less than 5000, how many integers could Claire have chosen?

(A) 1548 **(B)** 1584 **(C)** 1620 **(D)** 1656 **(E)** 1692

Proposed by ihatemath123

We perform casework on how many distinct digits are in N. Claire will only have at least 12 integers if the digits of N are distinct, or if exactly one of the digits of N appears twice, yielding 24 and 12 permutations, respectively. If there a digit appears three or more times, or there are two or more sets of equal digits in N, there will be less than 12 distinct permutations of the digits of N. Thus, there are only two cases.

1. If all four digits are distinct, let the digits of N be a, b, c and d, in some order, where $1 \leq a < b < c < d$. Then, Claire will write down exactly 6 numbers with a thousands digit of each of a, b, c and d. The smallest 12 integers she writes start with a or b, and the rest of the numbers start with c or d. So, if the smallest 12 integers are the only integers less than 5000, we have that

$$a < b < 5 \text{ and } 5 \leq c < d.$$

This leaves us with $\binom{4}{2} = 6$ ways to choose a and b, and $\binom{5}{2} = 10$ ways to choose c and d, so there are 60 ways to choose a, b, c and d. Furthermore, there are $4! = 24$ ways to permute these digits, so in this case, there are $60 \cdot 24 = 1440$ possible values of N.

2. If exactly one of the digits of N appears twice, let the digits of N be a, b, and c in some order, where a is the repeated digit. Since there are exactly 12 distinct permutations of the digits of N, each permutation must be an integer less than 5000, implying each of a, b, and c must each be less than 5. There are 4 ways to choose the repeated digit, 3 ways to choose the two non-repeated digits, and then 12 ways to permute the digits, so in this case, there are $4 \cdot 3 \cdot 12 = 144$ possible values of N.

Combining the two cases, the total number of possible values of N is $1440 + 144 = \boxed{\textbf{(B) } 1584}$.

Problem 18:

There exist (not necessarily distinct) digits X and Y such that when a positive integer is expressed in bases 10 and 11, the results are $336{,}149{,}XY3_{\text{ten}}$ and $162{,}82X{,}1Y5_{\text{eleven}}$, respectively. What digit is represented by X?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Proposed by ihatemath123

Using modulo 10, we have that

$$\begin{aligned} 3 \equiv\ & 336{,}149{,}XY3_{\text{ten}} \\ =\ & 162{,}82X{,}1Y5_{\text{eleven}} \\ =\ & (5) + (Y \cdot 11) + \left(1 \cdot 11^2\right) + \cdots + \left(6 \cdot 11^7\right) + \left(1 \cdot 11^8\right) \\ \equiv\ & 5 + Y + 1 + X + 2 + 8 + 2 + 6 + 1 \\ \equiv\ & 5 + X + Y \pmod{10} \\ & \implies \\ 3 \equiv\ & 5 + X + Y \pmod{10} \\ X + Y \equiv\ & 8 \pmod{10}. \end{aligned}$$

Also, using modulo 11,

$$\begin{aligned} 5 \equiv\ & 162{,}82X{,}1Y5_{\text{eleven}} \\ =\ & 336{,}149{,}XY3_{\text{ten}} \\ =\ & (3) + (Y \cdot 10) + \left(X \cdot 10^2\right) + \cdots + \left(3 \cdot 10^7\right) + \left(3 \cdot 10^8\right) \\ \equiv\ & (3) + (Y \cdot (-1)) + \left(X \cdot (-1)^2\right) + \cdots + \left(3 \cdot (-1)^7\right) + \left(3 \cdot (-1)^8\right) \\ \equiv\ & 3 - Y + X - 9 + 4 - 1 + 6 - 3 + 3 \\ \equiv\ & 3 - Y + X \pmod{11} \\ & \implies \\ 5 \equiv\ & 3 - Y + X \pmod{11} \\ X - Y \equiv\ & 2 \pmod{11}. \end{aligned}$$

Now, we have the system

$$\begin{cases} X - Y & \equiv 2 \pmod{11} \\ X + Y & \equiv 8 \pmod{10} \end{cases}$$

By inspection, the only pair of nonnegative integers (X, Y) that satisfies $X + Y = 8$ and the first congruence is $(X, Y) = (5, 3)$. The problem asks for X, which is $\boxed{\textbf{(E)}\ 5}$.[3]

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[3] Extra credit: when the positive integer mentioned in the problem is expressed in base thirty, what is the result? (Caculator allowed.)

Problem 19:

Leon randomly places two paper squares on a square surface such that:

- Each paper square lies completely within the square surface, and
- The sides of each paper square are parallel to those of the square surface.

If the square surface has a side length of 6 and the two paper squares have side lengths of 1 and 2, the probability that the squares overlap can be written as $\frac{m}{n}$ for coprime positive integers m and n. What is $m+n$?

(A) 461 **(B)** 521 **(C)** 562 **(D)** 569 **(E)** 610

Proposed by ihatemath123

Suppose the coordinates of the bottom left hand corner of the paper squares with side lengths 1 and 2 are (x_1, y_1) and (x_2, y_2), respectively. Then, x_1 and y_1 are randomly chosen real numbers between 0 and 5, and x_2 and y_2 are randomly chosen real numbers between 0 and 4.

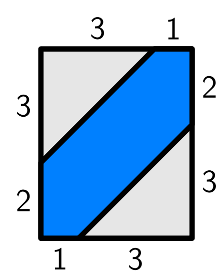
In order for the squares to overlap, the randomly chosen x_1 and x_2 values must be sufficiently close enough to each other, and the y_1 and y_2 values also be sufficiently close enough to each other. In particular, the squares overlap iff

$$-1 < x_1 - x_2 < 2 \text{ and } -1 < y_1 - y_2 < -2.$$

These two events are independent, and have the same probability of occurring, so we will calculate the probability that $-1 < x_1 - x_2 < 2$, and square it to get our final answer.

(This problem has now been reduced to the classical situation where two people randomly pick times to arrive at some meeting location.)

Since $0 < x_1 < 5$ and $0 < x_2 < 4$, we can think of randomly picking a point in a 5 by 4 rectangle corresponding to an ordered pair (x_1, x_2). We want to find the probability that $-1 < x_1 - x_2 < 2$. When graphed, the region satisfying this property is just the region between two parallel lines. The region of success in the 5 by 4 rectangle is shown below:



The region of success is everything in the 5 by 4 rectangle that is not grayed out; the area of the region grayed out is that of two triangles with area 4.5, for a total of 9; therefore, the area of the region of success is $20 - 9 = 11$. So, the probability that $-1 < x_1 - x_2 < 2$ is $\frac{11}{20}$, so the probability that the squares overlap is

$$\frac{11^2}{20^2} = \frac{121}{400}.$$

The problem asks for the sum of the numerator and denominator, which is $\boxed{\textbf{(B)}\ 521}$.

Problem 20:

In tetrahedron $ABCD$, faces $\triangle BCD$, $\triangle CAD$ and $\triangle ABD$ are acute and face $\triangle ABC$ is equilateral. The distances from D to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ and plane ABC are 13, 15, 20 and 12, respectively. What is the surface area of $ABCD$?

(A) $680\sqrt{3}$ **(B)** $700\sqrt{3}$ **(C)** $720\sqrt{3}$ **(D)** $760\sqrt{3}$ **(E)** $780\sqrt{3}$

Proposed by ihatemath123

[4]Let X, Y and Z be the feet from D to $\overline{BC}$, $\overline{AC}$ and $\overline{AB}$, respectively. Let P be the foot from D to plane ABC. Because we're given that all the faces are acute, P must lie inside $\triangle ABC$.

Claim: $\overline{PX} \perp \overline{BC}$, $\overline{PY} \perp \overline{AC}$ and $\overline{PZ} \perp \overline{AB}$.

This is essentially true by intuition; in case that's not convincing, note that

$$BP^2 + PD^2 = BD^2 = BX^2 + XD^2 = BX^2 + XP^2 + PD^2 \implies BX^2 + XP^2 = BP^2,$$

which means, by the converse of the Pythagorean theorem, that $\angle BXP = 90^\circ$. With similar logic on points Y and Z, we have shown the claim.

Using Pythagorean Theorem on $\triangle PXD$, $\triangle PYD$ and $\triangle PZD$, we get that $PX = 5$, $PY = 9$, and $PZ = 16$.

Since $\triangle ABC$ is equilateral, let s be its side length. Computing the area of $\triangle ABC$ in two ways,

$$\frac{s^2\sqrt{3}}{4} = [ABC] = [ABP] + [BCP] + [CAP] = \frac{1}{2} \cdot 5s + \frac{1}{2} \cdot 9s + \frac{1}{2} \cdot 16s \implies s = 20\sqrt{3}.$$

Thus, the surface area of $ABCD$ is

$$[ABC] + [ABD] + [BCD] + [CAD] = \frac{s^2\sqrt{3}}{4} + \frac{1}{2} \cdot 13s + \frac{1}{2} \cdot 15s + \frac{1}{2} \cdot 20s = \boxed{\textbf{(E) } 780\sqrt{3}}.$$

[4]lol im too lazy to figure out how 3d asymptote works - ihatemath123

Problem 21:

Let x and y be reals such that

$$\begin{cases} x\lfloor y\rfloor + y\lfloor x\rfloor &= 66 \\ x\lfloor x\rfloor + y\lfloor y\rfloor &= 144. \end{cases}$$

The value xy can be written as $\frac{p}{q}$ for coprime positive integers p and q. What is $p+q$? (Here, $\lfloor x\rfloor$ denotes the greatest integer not exceeding x.)

(A) 281 **(B)** 282 **(C)** 283 **(D)** 284 **(E)** 285

Proposed by ihatemath123

Adding and subtracting these equations, we obtain the system:

$$\begin{cases} x\lfloor y\rfloor + x\lfloor x\rfloor + y\lfloor x\rfloor + y\lfloor y\rfloor &= 210 \\ x\lfloor x\rfloor - x\lfloor y\rfloor + y\lfloor y\rfloor - y\lfloor x\rfloor &= 78. \end{cases}$$

We can factor the LHS of both equations:

$$\begin{cases} (x+y)(\lfloor x\rfloor + \lfloor y\rfloor) &= 210 \\ (x-y)(\lfloor x\rfloor - \lfloor y\rfloor) &= 78. \end{cases}$$

Using the properties of floors, we have that $\lfloor x\rfloor + \lfloor y\rfloor \le x+y < \lfloor x\rfloor + \lfloor y\rfloor + 2$ and $\lfloor x\rfloor - \lfloor y\rfloor - 1 < x-y < \lfloor x\rfloor - \lfloor y\rfloor + 1$. Since both $\lfloor x\rfloor + \lfloor y\rfloor$ and $\lfloor x\rfloor - \lfloor y\rfloor$ are integers, we conclude that

$$\lfloor x\rfloor + \lfloor y\rfloor \in \{-15, 14\}$$
$$\lfloor x\rfloor - \lfloor y\rfloor \in \{-9, 9\}.$$

Due to parity, we can further deduce that $\lfloor x\rfloor + \lfloor y\rfloor = -15$. Narrowing down the value of $\lfloor x\rfloor - \lfloor y\rfloor$ is irrelevant, since the two values correspond to the result when x and y are swapped with each other. For the rest of this solution, we will assume that $x > y$, so $\lfloor x\rfloor - \lfloor y\rfloor = 9$. So, we have that

$$\begin{cases} x+y &= \frac{210}{-15} = -14 \\ x-y &= \frac{78}{9} = \frac{26}{3}. \end{cases}$$

Solving this linear system, we have $x = -\frac{8}{3}$ and $y = -\frac{34}{3}$, so $xy = \frac{272}{9}$. Our answer extraction asks for $272 + 9 = \boxed{\textbf{(A) } 281}$.

Problem 22:

In a school with 250 students, each student belongs in exactly one of three friend groups, and each friend group contains at least two members. When each student shakes hands once with everybody that belongs in their friend group, it is found that exactly half of the total handshakes are made by one particular friend group with N students. What is the sum of the digits of N?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Proposed by ihatemath123

Let a and b be the number of students in the other two friend groups: we want to find $N = 250 - a - b$. We have that

$$\begin{aligned}\binom{a}{2} + \binom{b}{2} &= \binom{250-a-b}{2}. \\ a(a-1) + b(b-1) &= (250-a-b)(249-a-b) \\ a^2 + b^2 - a - b &= 250 \cdot 249 + a^2 + b^2 - 499a - 499b + 2ab \\ -2ab + 498a + 498b &= 250 \cdot 249 \\ ab - 249a - 249b &= -125 \cdot 249 \\ (a-249)(b-249) &= 124 \cdot 249 \\ (249-a)(249-b) &= 124 \cdot 249.\end{aligned}$$

The RHS can be prime factorized as $2^2 \times 3 \times 31 \times 83$. Since we have the bounds that $1 < a, b < 249$, we deduce that $249 - a = 186$ and $249 - b = 166$, or vice versa. So, $a = 63$ and $b = 83$. The problem asks for the sum of the digits of $250 - a - b = 104$, which is $\boxed{\textbf{(A) } 5}$.

Problem 23:

Twenty five points are evenly spaced around a circle. Among the 300 chords connecting two of these points, how many ways can two distinct chords be chosen and extended into lines, such that the two lines intersect and that the angle formed by their intersection is an integer?

(A) 6150 **(B)** 6300 **(C)** 6600 **(D)** 7200 **(E)** 8450

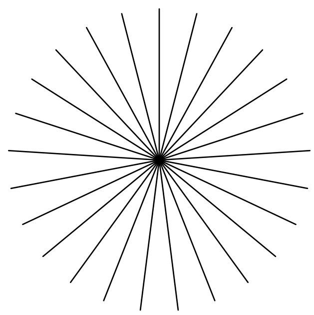
Proposed by ihatemath123

(For rigorousness, this proof uses the term "vector", but linear algebra is not actually used here.)

These twenty five points are the vertices of a regular 25-gon, and the 300 chords are the diagonals and sides of this 25-gon. Note that in a k-gon, for odd k, each diagonal is always parallel to some side of the polygon. For each side of a 25-gon, $\frac{300}{25} = 12 - 1 = 11$ diagonals are parallel to it.

We may direct each side/diagonal such that **a)** no two sides of the polygon point towards each other, and **b)** any two parallel sides/diagonals point in the same direction. Then, let's treat the 300 sides and diagonals as vectors, using the directions we assigned.

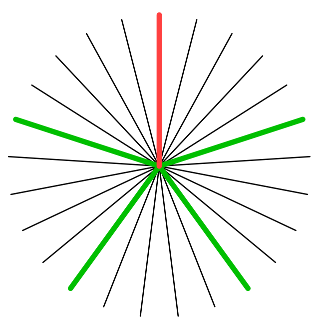
These vectors will point in 25 different directions, and 12 vectors will point in each direction:



Treating the sides/diagonals as vectors does not affect whether the angle formed by their intersection is an integer; in particular, iff two sides/diagonals intersected at an integer angle before rearranging as vectors, the angle between their vectors will also be an integer.

The angle between any two vectors that are adjacent rotationally is $\frac{360}{25}^\circ = \frac{72}{5}^\circ$.

For any one of these 300 vectors, exactly $4 \cdot 12$ vectors exist such that the angle formed between these two vectors is a (nonzero) integer. See the diagram below:



If we pick a vector pointing in the red direction, any vector pointing in a green direction will form an integer angle with it. (in particular, an angle of $\left(\frac{72}{5} \cdot 5\right)^\circ$ or $\left(\frac{72}{5} \cdot 10\right)^\circ$.) There are four directions this second vector could point in and 12 vectors pointing in each direction, so there are a total of 48 vectors we could choose.

In other words, for any of the 300 chords we could pick, there always exist 48 chords whose intersection angle is an integer.

There are 300 ways to pick our first chord and 48 ways to pick the second; we have to divide by 2 because we overcounted every combination once. So, our final answer is

$$\frac{300 \cdot 48}{2} = \boxed{\textbf{(D) } 7200}.$$

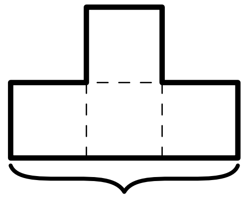
Problem 24:

Mulan has an infinite supply of T-shaped tetrominoes, shown below. Let N be the number of ways to tile a 4 inch by 24 inch rectangle with her tetrominoes. What is the sum of the digits of N? (Here, Mulan's tetrominoes are constructed out of four unit squares.)

(A) 10 **(B)** 12 **(C)** 14 **(D)** 16 **(E)** 18

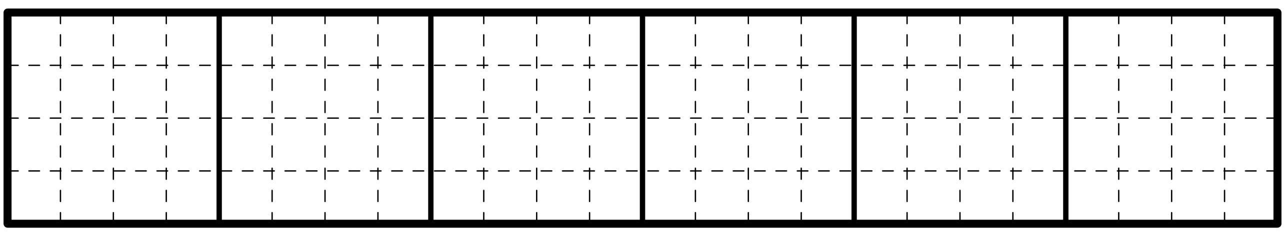
Proposed by ihatemath123

Mulan's tetromino:

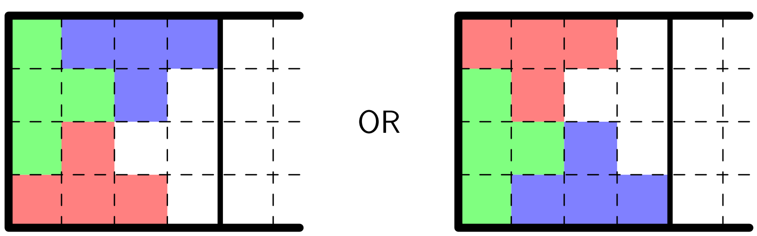


(Note that in the diagrams, tetrominoes will be colored in different colors; these are just for decoration/visibility, and have no meaning in terms of explaining the solution.)

For this solution, it helps to think of the 4 inch by 24 inch rectangle as six adjacent 4 inch by 4 inch squares:

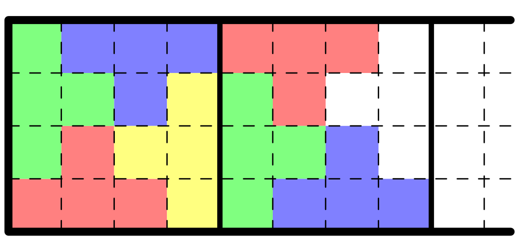


It turns out that because of how narrow the rectangle is, we're quite limited in the number of ways to tile it. To start off, we are forced to cover up the left side of the rectangle in one of the two following ways:



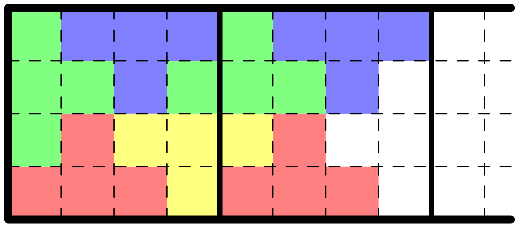
(Let's refer to these choices as "B1" and "B2" for **B**eginning the square) These two cases are just vertical reflections of each other. Regardless of which one we choose, we can either:

- Fill in the remaining gap in that 4 by 4 square with a tetromino; now, we have a 20 by 4 rectangle remaining, and we have **two choices** as to how to fill in the left side of it. When this is done, we will be left in the exact same scenario as before, but with a rectangle that is 4 squares narrower. Below is one of the two choices we could make:



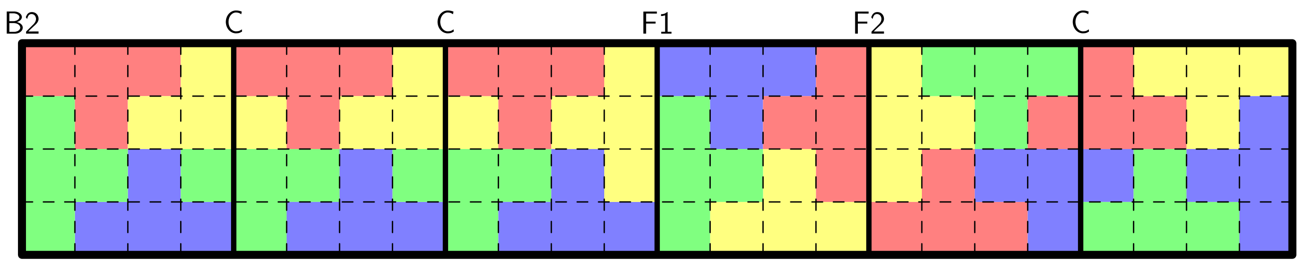
(Let's refer to these choices as "F1" and "F2" for **F**inishing the square.)

- We don't necessarily have to close off that square. After picking one of the initial orientations, we could make the **single choice** to continue in the following way:



(Let's refer to this choice as "C" for **C**ontinuing onwards.)

So, after choosing one of our 2 choices for how to orient the left side of the rectangle, we then have $2+1=3$ choices for how to continue. Notice that regardless of which of these three choices we choose, we will be left with the same "shape" to fill in; we now have three choices again! This repeats and repeats, until we have filled in our entire square. Below is an example of a filled in rectangle:



At the beginning, we either choose the choice of B1 or B2; after that, we have to make five choices between C, F1 or F2. Therefore, the number of ways to tile a 4 by 24 rectangle is

$$2 \cdot 3^5 = 486,$$

so our answer is $4+8+6 = \boxed{\textbf{(E) } 18}$.

Problem 25:

Define a sequence recursively by $a_1 = 1$ and

$$a_n = \begin{cases} 2 \times a_{(n-1)} & \text{for odd } n \\ 1 + a_{(n/2)} & \text{for even } n \end{cases}$$

for integer values of n greater than 1. Find the units digit of $a_1 + a_2 + a_3 + \cdots + a_{1024}$.

(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

Proposed by ihatemath123

The fact that the problem asks us to find the sum up to a_{1024} is kind of "sus"; alongside the nature of the recursion (doubling the indices or adding one), this motivates us to define a function:

$$f(n) = a_1 + a_2 + a_3 + \cdots + a_{2^n}.$$

We wish to find $f(10)$. We will find a recursive formula for $f(n)$.

Observe that

$$\begin{aligned} & a_2 + a_4 + a_6 + \cdots + a_{2k} \\ = \ & a_1 + 1 + a_2 + 1 + a_3 + 1 + \cdots + a_k + 1 \\ = \ & a_1 + a_2 + a_3 + \cdots + a_k + k. \end{aligned}$$

Also, observe that

$$\begin{aligned} & a_1 + a_3 + a_5 + \cdots + a_{2k-1} \\ = \ & 1 + a_3 + a_5 + \cdots + a_{2k-1} \\ = \ & 1 + 2a_2 + 2a_4 + \cdots + 2a_{2k-2} \\ = \ & 1 + 2(a_2 + a_4 + \cdots + a_{2k-2}) \\ = \ & 1 + 2(a_1 + a_2 + \cdots + a_{k-1} + k - 1) \\ = \ & 2k - 1 + 2(a_1 + a_2 + \cdots + a_k) - 2a_k. \end{aligned}$$

Combining these two identities, we have that

$$\begin{aligned} & a_1 + a_2 + a_3 + \cdots + a_{2k} \\ = \ & (a_1 + a_3 + \cdots + a_{2k-1}) + (a_2 + a_4 + \cdots + a_{2k}) \\ = \ & (2k - 1 + 2(a_1 + a_2 + \cdots + a_k) - 2a_k) + (a_1 + a_2 + \cdots + a_k + k) \\ = \ & 3k - 1 - 2a_k + 3(a_1 + a_2 + \cdots + a_k). \end{aligned}$$

So, we have that

$$f(n) = 3 \cdot 2^{n-1} - 1 - 2a_{2^{n-1}} + 3f(n-1).$$

By directly applying the recurrence for a, we have that for integers k, $a_{2^k} = k + 1$. So, we can simplify our equation to

$$f(n) = 3 \cdot 2^{n-1} - 1 - 2n + 3f(n-1).$$

Now we just plug and chug, using the fact that f(0) = 1:

$$\begin{aligned}
f(0) &= 1 \\
f(1) &= 3 \cdot 2^0 - 1 - 2 \cdot 1 + 3f(0) = 3 \\
f(2) &= 3 \cdot 2^1 - 1 - 2 \cdot 2 + 3f(1) = 10 \\
f(3) &= 3 \cdot 2^2 - 1 - 2 \cdot 3 + 3f(2) = 35 \\
f(4) &= 3 \cdot 2^3 - 1 - 2 \cdot 4 + 3f(3) = 120 \\
f(5) &= 3 \cdot 2^4 - 1 - 2 \cdot 5 + 3f(4) = 397 \\
f(6) &= 3 \cdot 2^5 - 1 - 2 \cdot 6 + 3f(5) = 1274 \\
f(7) &= 3 \cdot 2^6 - 1 - 2 \cdot 7 + 3f(6) = 3999 \\
f(8) &= 3 \cdot 2^7 - 1 - 2 \cdot 8 + 3f(7) = 12364 \\
f(9) &= 3 \cdot 2^8 - 1 - 2 \cdot 9 + 3f(8) = 37841 \\
f(10) &= 3 \cdot 2^9 - 1 - 2 \cdot 10 + 3f(9) = 115038.
\end{aligned}$$

The answer is the units digit of $f(10)$, which is $\boxed{\textbf{(E) } 8}$.

(Of course, we did not directly need to compute $f(10)$; once we got this recursion, it would've sufficed to find the units digit of $f(k)$ for $k = 1, 2, \ldots, 10$. However, it was included in this solution just to show that we can directly compute the sum.)